Chapter 19

State-space control design: from open loop to feedback

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Leveraging the ODE simulation methods presented in §10, the numerical optimization methods discussed in §15, and the state-space representations of dynamic systems and their feedback control discussed in §17 and §18, the stage is now set to:

a) optimize a distribution of control inputs to guide a nonlinear ODE system along some trajectory to achieve a desired objective, while

b) coordinating corrections to these control inputs based on limited real-time (albeit, noisy) measurements of the system to keep the system close to this optimized trajectory despite unmodeled state disturbances and errors in the system model itself.
This chapter explains the mathematical foundation for accomplishing these tasks, and illustrates with some representative examples.

To focus the presentation, it is instructive to first have a representative problem in mind. Thus, though the method we will present has broad applicability and is easily generalized in a variety of ways, consider specifically, for the moment, the double inverted pendulum swing up and stabilization problem illustrated in Figure 19.1. In this problem, there are two pendula of different lengths hanging from a cart; each pendulum rotates freely about the end attached to the cart. One pendulum hangs behind the track and the other hangs in front, so there is no interference between the two. We can measure the angles \( \theta_1, \theta_2 \) of the pendula, measured from upright, and the position \( x \) of the cart, measured from the center of the track; the goal is to bring the system to the unstable equilibrium state \( \{ \theta_1, \theta_2, x, \dot{\theta}_1, \dot{\theta}_2, \dot{x} \} = \{0,0,0,0,0,0\} \) and stabilize it there. We will return to this problem in §19.1.5.1, immediately after we have introduced the tools to solve it.

### 19.1 Summary of the continuous-time case

An introduction to continuous-time state-space control and estimation strategies, via both iterative adjoint-based optimization and direct Riccati-based feedback, is now presented. The presentation is divided into four parts: §19.1.1 and 19.1.2 consider the **control problem** (i.e., the determination of appropriate inputs to a system to achieve a desired objective assuming accurate knowledge of the system state), whereas §19.1.3 and 19.1.4 consider the **estimation problem** (i.e., the approximate determination of the system state based on recent, limited, noisy measurements of an actual physical system). Denoting the current time as \( t = 0 \), the control problem looks at the future evolution of the system over a horizon of interest \([0,T]\), whereas the estimation problem looks at the past history of measurements from the system over a horizon of interest \([-T,0]\). Together, solutions of the control and estimation problems facilitate the coordination of a limited number of actuators with a limited number of sensors in order to achieve a desired effect (§19.1.5).

The **iterative approach** to these two problems (see §19.1.1 and 19.1.3) is applicable to both nonlinear
systems and non-quadratic cost functions. Significantly, it only requires the computation of vectors (i.e., state and adjoint fields) of dimension $N$ (the dimension of the state field itself) evolving over the (finite) time horizon of interest, and thus extends readily to high-dimensional discretizations of unsteady PDE systems, even when $N \gtrsim 10^6$ is necessary to resolve the system under consideration. Essentially, for any smooth, differentiable system one can afford to simulate computationally, one can also afford to simulate the adjoint field necessary to determine the gradient of the cost function of interest in the space of the optimization variables, thereby enabling gradient-based optimization.

The direct approach to these problems (§19.1.2 and 19.1.4) is based on more strict assumptions [specifically, a linear governing equation and a quadratic cost function]. Subject to these assumptions, this approach jumps straight to the unique minimum of the cost function by setting the gradient equal to zero and solving the two-point boundary value problem for the state and adjoint fields that results. This approach requires the computation of a matrix (relating the state and adjoint fields in the optimal solution) of dimension $N^2$. An efficient technique to calculate this matrix equation at the heart of these problems in the continuous-time, infinite-horizon, linear time invariant (LTI) case is discussed in §4.5.2. Such matrix-based approaches do not extend readily to high-dimension discretizations of infinite-dimensional PDE systems, as they are prohibitively expensive for $N \gtrsim 10^3$.

The adjoint-based control optimization approach (§19.1.1) is known as model predictive control. The Riccati-based feedback control approach (§19.1.2) is known as $H_2$ state feedback control or optimal control. The adjoint-based state estimation approach (§19.1.3) is known (in weather forecasting) as 4Dvar. The Riccati-based state estimation approach (§19.1.4) is known as a Luenberger observer or, when interpreted from a stochastic point of view (see §20.1 and 20.2), as $H_\infty$ state estimation or a Kalman filter.

### 19.1.1 Control via adjoint-based iterative optimization

We assume the system of interest is governed by a continuous-time state equation of the form

$$
E \frac{d\mathbf{x}}{dt} = N(\mathbf{x}, \mathbf{f}, \mathbf{u}) \quad \text{on } 0 < t < T,
$$

$$
\mathbf{x} = \mathbf{x}_0 \quad \text{at } t = 0,
$$

where $t = 0$ is the present time and

- $\mathbf{x}(t)$ is the state vector with $\mathbf{x}_0$ the (known) initial condition (at $t = 0$),
- $\mathbf{f}(t)$ is the (known) applied external force (e.g., gravity), and
- $\mathbf{u}(t)$ is the “control” (e.g., some force on the system that we may prescribe).

The matrix $E$, which may be singular, and the nonlinear function $N(\mathbf{x}, \mathbf{f}, \mathbf{u})$ may be defined as necessary in order to represent any smooth ODE of interest, including both low-dimensional ODEs and high-dimensional discretizations of PDE systems. We also define a cost function $J$ which measures any trajectory of this system such that

$$
J = \frac{1}{2} \int_0^T \left[ |\mathbf{x}(t)|_{Q_x}^2 + |\mathbf{u}(t)|_{Q_u}^2 \right] dt + \frac{1}{2} |E\mathbf{x}(T)|_{Q_T}^2.
$$

The norms are each weighted such that, e.g., $|\mathbf{x}(t)|_{Q_x}^2 \triangleq \mathbf{x}^T Q_x \mathbf{x}$, with $Q_x \geq 0$, $Q_u > 0$, and $Q_T \geq 0$. The cost function (specifically, the selection of $Q_x$, $Q_u$, and $Q_T$) represents mathematically what we would like the controls $\mathbf{u}$ to accomplish in this system\(^1\). In short, the problem at hand is to minimize $J$ with respect to the control distribution $\mathbf{u}(t)$ subject to (19.1).

\(^1\)Physically, we can say that the control objective is to minimize some measure of the state [as measured by the first and third terms of $J$ in (19.1c)] without using too much control effort to do it [as measured by the second term of $J$]. Nonquadratic forms for $J$ are also possible. Note in particular that the terminal cost [the last term of (19.1c)] enables, in effect, the penalization of the dynamics likely to come after the finite optimization horizon $t \in [0, T]$; including such a term in the optimization problem significantly improves its long time behavior when applied in the receding-horizon model predictive control framework (Bitmead et al. 1990).
We now consider what happens when we simply perturb the inputs to our original system (19.1) a small amount. Small perturbations \( \mathbf{u'} \) to the control \( \mathbf{u} \) cause small perturbations \( \mathbf{x'} \) to the state \( \mathbf{x} \). Such perturbations are governed by the perturbation equation, a.k.a. tangent linear equation,

\[
\mathcal{L}\mathbf{x'} = \mathbf{B} \mathbf{u'} \leftrightarrow \frac{d\mathbf{x'}}{dt} = A\mathbf{x'} + \mathbf{B} \mathbf{u'} \quad \text{on } 0 < t < T, \tag{19.2a}
\]

\[
\mathbf{x'} = 0 \quad \text{at } t = 0, \tag{19.2b}
\]

where the operator \( \mathcal{L} = (E \frac{d}{dt} - A) \) and matrices \( A \) and \( B \) are obtained via the linearization\(^2\) of (19.1) about the trajectory \( \mathbf{x}(\mathbf{u}) \). The concomitant small perturbation to the cost function \( J \) is given by

\[
J' = \int_0^T \left( \mathbf{x}^H \mathbf{Q}_x \mathbf{x}' + \mathbf{u}^H \mathbf{Q}_u \mathbf{u}' \right) dt + \mathbf{x}^H(T) E^H \mathbf{Q}_T \mathbf{E} \mathbf{x}'(T). \tag{19.2c}
\]

Note that (19.2a) implicitly represents a linear relationship between \( \mathbf{x}' \) and \( \mathbf{u}' \). Knowing this, the task before us is to reexpress \( J' \) in such a way as to make the resulting linear relationship between \( J' \) and \( \mathbf{u}' \) explicitly evident, at which point the gradient \( \frac{\partial J}{\partial \mathbf{u}} \) may readily be defined. To this end, define the weighted inner product \( \langle \langle \mathbf{a}, \mathbf{b} \rangle \rangle \doteq \int_0^T \mathbf{a}^H \mathbf{b} \, dt \) and express the following adjoint identity

\[
\langle \langle \mathbf{r}, \mathcal{L} \mathbf{x}' \rangle \rangle = \langle \langle \mathcal{L} \mathbf{r}, \mathbf{x}' \rangle \rangle + \mathbf{b}. \tag{19.3}
\]

Using integration by parts, it follows that \( \mathcal{L}' \mathbf{r} = -(E \frac{d}{dt} + A^H) \mathbf{r} \) and \( \mathbf{b} = [\mathbf{r}^H \mathbf{E} \mathbf{x}']^T = 0 \). We now define the relevant adjoint equation by

\[
\mathcal{L}' \mathbf{r} = \mathbf{Q}_x \mathbf{x} \quad \leftrightarrow \quad -E \frac{d\mathbf{r}}{dt} = A^H \mathbf{r} + \mathbf{Q}_x \mathbf{x} \quad \text{on } 0 < t < T, \tag{19.4a}
\]

\[
\mathbf{r} = \mathbf{Q}_T \mathbf{E} \mathbf{x} \quad \text{at } t = T. \tag{19.4b}
\]

The adjoint field \( \mathbf{r} \) so defined is easy to compute via a backwards march from \( t = T \) back to \( t = 0 \). Both \( A^H \) and the forcing term \( \mathbf{Q}_x \mathbf{x} \) in (19.4a) are functions of \( \mathbf{x}(t) \), which itself must be determined from a forward march of (19.1) from \( t = 0 \) to \( t = T \); thus \( \mathbf{x}(t) \) must be saved on this forward march over the interval \( t \in [0, T] \) in order to calculate (19.4) via a backwards march from \( t = T \) back to \( t = 0 \). The need for storing \( \mathbf{x}(t) \) on \( [0, T] \) during this forward march in order to construct the adjoint on the backward march can present a significant storage problem. This problem may be averted with a checkpointing algorithm which saves \( \mathbf{x}(t) \) only occasionally on the forward march, then recomputes \( \mathbf{x}(t) \) as necessary from these “checkpoints” during the backward march for \( \mathbf{r}(t) \). Noting (19.2) and (19.4), it follows from (19.3) that

\[
\int_0^T \mathbf{r}^H \mathbf{B} \mathbf{u'} \, dt = \int_0^T \mathbf{x}^H \mathbf{Q}_x \mathbf{x}' \, dt + \mathbf{x}^H(T) E^H \mathbf{Q}_T \mathbf{E} \mathbf{x}'(T) \quad \Rightarrow \quad J' = \int_0^T \left[ \mathbf{B}^H \mathbf{r} + \mathbf{Q}_u \mathbf{u} \right]^H \mathbf{u}' \, dt \doteq \langle \langle \frac{\partial J}{\partial \mathbf{u}}, \mathbf{u}' \rangle \rangle.
\]

As \( \mathbf{u}' \) is arbitrary, the desired gradient is thus given by

\[
\frac{\partial J}{\partial \mathbf{u}} = \mathbf{B}^H \mathbf{r} + \mathbf{Q}_u \mathbf{u}, \tag{19.5}
\]

and is readily determined from the adjoint field \( \mathbf{r} \) defined by (19.4). This gradient may be used to update \( \mathbf{u} \) at each iteration \( k \) via any of a number of standard optimization strategies, including steepest descent, preconditioned nonquadratic conjugate gradient, and limited-memory BFGS (see §2?).

\(^2\)That is, substitute \( \mathbf{x} + \mathbf{x}' \) for \( \mathbf{x} \) and \( \mathbf{u} + \mathbf{u}' \) for \( \mathbf{u} \) in (19.1a), multiply out, and retain all terms that are linear in the perturbation quantities.
Note also that, at a given iteration \( k \) of a gradient-based optimization procedure, for a given value of the optimization variables \( \mathbf{u}^k \) and a given update direction \( \mathbf{p}^k \), one needs to determine the parameter of descent \( \alpha \) to perform a “line minimization,” that is, to minimize \( J(\mathbf{u}^k + \alpha \mathbf{p}^k) \) with respect to \( \alpha \). By solving the perturbation equation (19.2) for \( \mathbf{x}' \) in the direction \( \mathbf{u}' = \mathbf{p}^k \) from the point \( \mathbf{u} = \mathbf{u}^k \) and \( \mathbf{x} = \mathbf{x}^k \), it is straightforward to get an estimate of the most suitable value for \( \alpha \) in the case that \( J \) is nearly quadratic in \( \mathbf{u} \). Fixing \( \mathbf{u}^k \) and \( \mathbf{p}^k \) for the moment, performing a truncated Taylor series expansion for \( J(\mathbf{u}^k + \alpha \mathbf{p}^k) \) about the value \( J(\mathbf{u}^k) \), and setting the derivative with respect to \( \alpha \) to equal zero gives

\[
\alpha = -\frac{dJ(\mathbf{u}^k + \alpha \mathbf{p}^k)}{d\alpha} \bigg|_{\alpha=0}, \tag{19.6}
\]

where the derivatives shown are simple functions of \( \{\mathbf{x}, \mathbf{x}', \mathbf{u}, \mathbf{u}'\} \), as readily determined from (19.1c). This value of \( \alpha \) minimizes \( J(\mathbf{u}^k + \alpha \mathbf{p}^k) \) if \( J \) is quadratic in \( \mathbf{u} \). If it is not quadratic (for example, if the relationship \( \mathbf{x}(\mathbf{u}) \) implied by (19.1a) is nonlinear), this value of \( \alpha \) might not accurately minimize \( J(\mathbf{u}^k + \alpha \mathbf{p}^k) \) with respect to \( \alpha \), and in fact may lead to an unstable algorithm if used at each iteration of a gradient descent procedure. However, (19.6) is still useful to initialize the guess for \( \alpha \) at each iteration.

### 19.1.2 Control via Riccati-based feedback

By (19.5), the control \( \mathbf{u} \) which minimizes \( J \) is given by

\[
\frac{\partial J}{\partial \mathbf{u}} = 0 \quad \Rightarrow \quad \mathbf{u} = -\mathbf{Q}^{-1} \mathbf{B}^H \mathbf{r}. \tag{19.5}
\]

We now consider the problem that arises when we start with a continuous-time governing equation (19.1a)-(19.1b) for the state variable \( \mathbf{x}(t) \) that is already in the linearized form of a perturbation equation [as in (19.2a)-(19.2b)]. In other words, we perturb the (already linear) system about the control distribution \( \mathbf{u} = 0 \) and the trajectory \( \mathbf{x}(\mathbf{u}) = 0 \), and thus the perturbed system is \( \mathbf{u} = \mathbf{u}' \) and \( \mathbf{x} = \mathbf{x}' \). Combining the perturbation and adjoint equations (19.2) and (19.4) into a single matrix form, applying the optimal value of the control \( \mathbf{u} \) as noted above, and assuming for simplicity that \( E = I \), gives:

\[
\frac{d}{dt} \begin{bmatrix} \mathbf{x}' \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BQ}^{-1}\mathbf{B}^H \\ -\mathbf{Qx} & -\mathbf{A}^H \end{bmatrix} \begin{bmatrix} \mathbf{x}' \\ \mathbf{r} \end{bmatrix} \quad \text{where} \quad \begin{cases} \mathbf{x}' = 0 & \text{at } t = 0, \\
\mathbf{r} = \mathbf{Q}_T \mathbf{x}' & \text{at } t = T. \end{cases} \tag{19.7}
\]

This ODE, with both initial and terminal conditions, is referred to as a \textbf{two-point boundary value problem}. Its general solution may be found via the \textbf{sweep method} (Bryson & Ho 1969): assuming there exists a relation between the perturbation vector \( \mathbf{x}' = \mathbf{x}'(t) \) and the adjoint vector \( \mathbf{r} = \mathbf{r}(t) \) via a matrix \( \mathbf{X} = \mathbf{X}(t) \) such that

\[
\mathbf{r} = \mathbf{X} \mathbf{x}', \tag{19.8}
\]

inserting this assumed form of the solution (a.k.a. \textbf{solution ansatz}) into the combined matrix form (19.7) to eliminate \( \mathbf{r} \), combining rows to eliminate \( dx'/dt \), factoring out \( \mathbf{x}' \) to the right, and noting that this equation holds for all \( \mathbf{x}' \), it follows that \( \mathbf{X} \) obeys the \textbf{differential Riccati equation} (DRE)

\[
-\frac{d\mathbf{X}}{dt} = \mathbf{A}^H \mathbf{X} + \mathbf{X} \mathbf{A} - \mathbf{X} \mathbf{BQ}^{-1}\mathbf{B}^H \mathbf{X} + \mathbf{Q_x} \quad \text{where} \quad \mathbf{X}(T) = \mathbf{Q}_T, \tag{19.9a}
\]

where the condition at \( \mathbf{X}(T) \) follows from (19.7) and (19.8). Solutions \( \mathbf{X} = \mathbf{X}(t) \) of this matrix equation satisfy \( \mathbf{X}^H = \mathbf{X} \), and may easily be determined via marching procedures similar to those used to march ODEs (Crank-Nicholson, Runge-Kutta, etc.). By the characterization of the optimal point, we may now write the control \( \mathbf{u} \) as

\[
\mathbf{u} = \mathbf{Kx} \quad \text{where} \quad \mathbf{K} = -\mathbf{Q}^{-1}\mathbf{B}^H \mathbf{X}. \tag{19.9b}
\]
To recap, this value of $K$ minimizes

$$J = \frac{1}{2} \int_0^T \left[ x^H Q_x x + u^H Q_u u \right] dt + \frac{1}{2} x^H(T) Q_T x(T) \quad \text{where} \quad \frac{dx}{dt} = Ax + Bu.$$ (19.9c)

The matrix $K = K(t)$ is referred to as the optimal control feedback gain matrix, and is a function of the solution $X$ to (19.9a). This equation may be solved for linear time-varying (LTV) or linear time-invariant (LTI) systems based solely on knowledge of $A$ and $B$ in the system model and $Q_x$, $Q_u$, and $Q_T$ in the cost function (that is, the gain matrix $K$ may be computed offline). Alternatively, if we take the limit that $T \to \infty$ (that is, if we consider the infinite-horizon control problem) and the system is LTI, the matrix $X$ in (19.9a) may be marched to steady state. This steady state solution for $X$ satisfies the continuous-time algebraic Riccati equation (CARE)

$$0 = A^H X + XA - XBQ_u^{-1}B^H X + Q_x.$$ (19.10)

Efficient algorithms to solve this quadratic matrix equation are discussed in §4.5.2.

19.1.3 Estimation via adjoint-based iterative optimization

The derivation presented here is analogous to that presented in §19.1.1. We first write the state equation modeling the system of interest in ODE form:

$$E \frac{dx}{dt} = N(x, f, v, w) \quad \text{on} \quad -T < t < 0,$$

$$x = u \quad \text{at} \quad t = -T,$$ (19.11a)

where $t = 0$ is the present time and

- $x(t)$ is the state vector,
- $f(t)$ models the known external forcing,

and the quantities to be optimized are:

- $u$, representing the unknown initial condition of the model (at $t = -T$),
- $v$, representing the unknown constant parameters of the model, and
- $w(t)$, representing the unknown external inputs which we would like to determine.

We next write a cost function which measures the misfit of the available measurements $y(t)$ with the corresponding quantity in the computational model, $C \langle x(t) \rangle$, and additionally penalizes the deviation of the initial condition $u$ from any a priori estimate of the initial condition $\bar{u}$, the deviation of the parameters $v$ from any a priori estimate of the parameters $\bar{v}$, and the magnitude of the disturbance terms $w(t)$:

$$J = \frac{1}{2} \int_{-T}^0 \left[ Cx - y \right]_0^2 dt + \frac{1}{2} \left| u - \bar{u} \right|_Q^2 + \frac{1}{2} \left| v - \bar{v} \right|_Q^2 + \frac{1}{2} \int_{-T}^0 \left| w \right|_Q^2 dt.$$ (19.11c)

The norms are each weighted with positive semi-definite matrices such that, e.g., $|y|_Q^2 \triangleq y^H Q_y y$ with $Q_y \succeq 0$.

In short, the problem at hand is to minimize $J$ with respect to $\{u, v, w(t)\}$. subject to (19.11).

Small perturbations $\{u', v', w'(t)\}$ to $\{u, v, w(t)\}$ cause small perturbations $x'$ to the state $x$. Such perturbations are governed by the perturbation equation

$$\mathcal{L}x' = B_x v' + B_w w' \quad \Rightarrow \quad E \frac{dx'}{dt} = A x' + B_x v' + B_w w' \quad \text{on} \quad -T < t < 0,$$

$$x' = u' \quad \text{at} \quad t = -T,$$ (19.12a)

In the 4Dvar setting, such an estimate $\bar{u}$ for $x(-T)$ is obtained from the previously-computed forecast, and the corresponding term in the cost function is called the “background” term. The effect of this term on the time evolution of the forecast is significant and sometimes detrimental, as it constrains the update to $u$ to be small when, in some circumstances, a large update might be warranted.
where the operator $\mathcal{L} = (E \frac{d}{dt} - A)$ and the matrices $A, B_u, \text{ and } B_w$ are obtained via the linearization of (19.11a) about the trajectory $x(u)$. The concomitant small perturbation to the cost function $J$ is given by

$$J' = \int_{-T}^{0} (Cx - y)^H Q_s Cx' dt + (u - \bar{u})^H Q_u u' + (v - \bar{v})^H Q_v v' + \int_{-T}^{0} w^H Q_w w' dt. \tag{19.13}$$

Again, the task before us is to re-express $J'$ in such a way as to make the resulting linear relationship between $J'$ and $\{u', v', w(t)\}$ explicitly evident, at which point the necessary gradients may readily be defined. To this end, we define the inner product $\langle a, b \rangle \triangleq \int_{-T}^{0} a^H b dt$ and express the adjoint identity

$$\langle r, Lx' \rangle = \langle L^* r, x' \rangle + b. \tag{19.14}$$

Using integration by parts, it follows that $L^* r = -(E^H \frac{d}{dt} + A^H) r$ and $b = [r^H E x]^T_{t=0}$. Based on this adjoint operator, we now define an adjoint equation of the form

$$L^* r = C^H Q_s (Cx - y) \iff -E^H \frac{d}{dt} r = A^H r + C^H Q_s (Cx - y) \quad \text{on } -T < t < 0, \tag{19.15a}$$

$$r = 0 \quad \text{at } t = 0. \tag{19.15b}$$

Note again that the difficulty involved with numerically solving the ODE given by (19.15) via a backward march from $t = 0$ to $t = -T$ is essentially the same as the difficulty involved with solving the original ODE (19.11).

Finally, combining (19.12) and (19.15) into the identity (19.14) and substituting into (19.13), it follows that

$$J' = \left[ E^H r(-T) + Q_u (u - \bar{u}) \right]^H u' + \left[ \int_{-T}^{0} B_u^T r dt + Q_v (v - \bar{v}) \right]^H v' + \left[ \int_{-T}^{0} B_w^T r + Q_w w \right]^H w' dt$$

$$\triangleq \left[ \frac{\partial J}{\partial u} \right]_{S_u} + \left[ \frac{\partial J}{\partial v} \right]_{S_v} + \left[ \frac{\partial J}{\partial w} \right]_{S_w},$$

for some $S_u > 0, S_v > 0, \text{ and } S_w > 0$ where $\langle a, b \rangle_S \triangleq a^H S b$ and $\langle a, b \rangle_S \triangleq \int_{-T}^{0} a^H S b dt$, and thus

$$\frac{\partial J}{\partial u} = S_u^{-1} \left[ E^H r(-T) + Q_u (u - \bar{u}) \right], \quad \frac{\partial J}{\partial v} = S_v^{-1} \left[ \int_{-T}^{0} B_v^T r dt + Q_v (v - \bar{v}) \right], \quad \text{and}$$

$$\frac{\partial J}{\partial w} = S_w^{-1} \left[ B_w^T r(t) + Q_w w(t) \right], \quad \text{for } t \in [-T, 0]. \tag{19.16}$$

We have thus defined the gradient of the cost function with respect to the optimization variables $\{u, v, w(t)\}$ as a function of the adjoint field $r$ defined in (19.15), which, for any trajectory $x(u, v, w)$ of our original system (19.11), may easily be computed.

The estimation technique described above was developed in parallel, and largely independently, in the controls and weather forecasting communities. In the weather forecasting community, the technique is referred to as the 4D (space/time) variational method, or 4Dvar. In the controls community, the technique is referred to as Moving Horizon Estimation (MHE). MHE was developed with low dimensional ODE systems in mind; implementations of MHE typically search for a small time-varying “state disturbance” model error term in addition to the initial state of the system in order reconcile the measurements with the model over the period of interest as accurately as possible. 4Dvar, in contrast, was developed with high dimensional discretizations of infinite-dimensional (PDE) systems in mind; in order to retain numerical tractability, implementations of 4Dvar typically do not search for such a time-varying model error term.

Another technique that has been introduced to accelerate MHE/4Dvar implementations is multiple shooting. With this technique, the horizon of interest is split into two or more subintervals. The initial conditions
(and, in some implementations, the time-varying model error term) for each subinterval are first initialized and optimized independently, then these several independent solutions are adjusted so that the trajectories coincide at the matching points between the subintervals.

19.1.4 Estimation via Riccati-based feedback

We now convert the Riccati-based estimation problem into an equivalent control problem of the form already solved (in §19.1.2). Consider the linear equations for the state $\mathbf{x}$, the state estimate $\hat{\mathbf{x}}$, and the state estimation error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$:

\[
\frac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u}, \quad y = C\mathbf{x}. \tag{19.17}
\]
\[
\frac{d\hat{\mathbf{x}}}{dt} = A\hat{\mathbf{x}} + B\hat{\mathbf{u}} - L(y - \hat{y}), \quad \hat{y} = C\hat{\mathbf{x}}. \tag{19.18}
\]
\[
\frac{d\tilde{\mathbf{x}}}{dt} = A\tilde{\mathbf{x}} + L\tilde{y}, \quad \tilde{y} = C\tilde{\mathbf{x}}. \tag{19.19}
\]

The output injection term $L(y - \hat{y})$ applied to the equation for the state estimate $\hat{\mathbf{x}}$ is to be designed to “nudge” this equation appropriately based on the available measurements $y$ of the actual system. If this term is doing its job correctly, $\tilde{\mathbf{x}}$ is driven towards $\mathbf{x}$ (that is, $\tilde{\mathbf{x}}$ is driven towards zero) even if the initial condition on $\mathbf{x}$ is unknown and (19.17) is only an approximate model of reality. For convenience and without loss of generality, we now return our focus to the time interval $[0,T]$ rather than the interval $[-T,0]$.

We thus set out to minimize some measure of the state estimation error $\tilde{\mathbf{x}}$ by appropriate selection of $L$. To this end, taking $\tilde{\mathbf{x}}^H$ times (19.19), we obtain

\[
\tilde{\mathbf{x}}^H \left[ \frac{d\tilde{\mathbf{x}}}{dt} = A\tilde{\mathbf{x}} + L\tilde{y} \right] = \frac{1}{2} \frac{d(\tilde{\mathbf{x}}^H\tilde{\mathbf{x}})}{dt}. \tag{19.20}
\]

Motivated by the second relation in brackets above, consider a new system

\[
\frac{d\mathbf{z}}{dt} = \tilde{A}\mathbf{z} + L\mathbf{u} \quad \text{where} \quad \mathbf{u} = L^H\mathbf{z} \quad \text{and} \quad \mathbf{z}(0) = \tilde{\mathbf{x}}(0). \tag{19.21}
\]

Though the dynamics of $\tilde{\mathbf{x}}(t)$ and $\mathbf{z}(t)$ are different, the evolution of their energy is the same, by (19.20). That is, $\tilde{\mathbf{x}}^H\tilde{\mathbf{x}} = \mathbf{z}^H\mathbf{z}$ even though, in general, $\mathbf{z}(t) \neq \tilde{\mathbf{x}}(t)$. We will thus, for convenience, design $L$ to minimize a cost function related to this auxiliary variable $\mathbf{z}$, defined here such that, taking $Q_1 = I$,

\[
J = \frac{1}{2} \int_0^T \left[ \mathbf{z}^H Q_1 \mathbf{z} + \mathbf{u}^H Q_2 \mathbf{u} \right] dt + \frac{1}{2} \tilde{A}^H (-T) P_0 \mathbf{z} (-T), \tag{19.22a}
\]

where, renaming $\bar{A} = A^H$, $\bar{B} = C^H$, and $\bar{K} = L^H$, (19.21) may be written as

\[
\frac{d\mathbf{z}}{dt} = \bar{A}\mathbf{z} + \bar{B}\mathbf{u} \quad \text{where} \quad \mathbf{u} = \bar{K}\mathbf{z}. \tag{19.22b}
\]

Finding the feedback gain matrix $\bar{K}$ in (19.22b) that minimizes the cost function $J$ in (19.22a) is exactly the same problem that is solved in (19.9), just with different variables. Thus, the optimal gain matrix $L$ which minimizes a linear combination of the energy of the state estimation error, $\tilde{\mathbf{x}}^H\tilde{\mathbf{x}}$, and some measure of the estimator feedback gain $L$ is again determined from the solution $P$ of a Riccati equation which, making the appropriate substitutions into the solution presented in (19.9), is given by

\[
\frac{dP}{dt} = AP + PA^H - PC^H Q_2^{-1} CP + Q_1, \quad P(0) = P_0, \quad L = -PC^H Q_2^{-1}. \tag{19.23}
\]

The compact derivation presented above gets quickly to the Riccati equation for an optimal estimator, but as the result of a somewhat contrived optimization problem. A more intuitive formulation is to replace the state equation (19.17) with

\[
\frac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u} + \mathbf{w}_1, \quad y = C\mathbf{x} + \mathbf{w}_2. \tag{19.24}
\]
where \( w_1 \) (the “state disturbance”) and \( w_2 \) (the “measurement noise”) are assumed to be uncorrelated, zero mean, white Gaussian processes with modeled covariance \( \mathbb{E}\{w_1 w_1^H\} = Q_1 \) and \( \mathbb{E}\{w_2 w_2^H\} = Q_2 \) respectively. As shown in §20.1, going through the necessary steps to minimize the expected energy of the estimation error, \( \mathbb{E}\{\hat{x}^H \hat{x}\} = \text{trace}(P) \) where \( P = \mathbb{E}\{\hat{x}\hat{x}^H\} \), we again arrive at an estimator of the form given in (19.18) with the feedback gain matrix \( L \) as given by (19.23).

### 19.1.5 The separation principle: putting it together

In §19.1.2, a convenient feedback relationship was derived for determining optimal control inputs based on full state information. In §19.1.4, a convenient feedback relationship was derived for determining an optimal estimate of the full state based on the available system measurements. It might seem like a good idea, then, to combine the results of these two sections: that is, in the practical case in which control inputs must be determined based on available system measurements, to develop an estimate of the state \( \hat{x} \) based on the results of §19.1.4, then to apply control \( u = K\hat{x} \) based on this state estimate and the results of §19.1.2. This is in fact a good idea, and the reason why is called the separation principle. Collecting the equations presented previously and adding a reference control input \( r \), we have

<table>
<thead>
<tr>
<th>Plant</th>
<th>( \dot{x}/dt = Ax + Bu + w_1 ), ( y = Cx + w_2 ),</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimator</td>
<td>( \dot{\hat{x}}/dt = A\hat{x} + Bu - L(\hat{y} - \hat{\hat{y}}) ), ( \hat{\hat{y}} = C\hat{x} ),</td>
</tr>
<tr>
<td>Controller</td>
<td>( u = K\hat{x} + r ),</td>
</tr>
</tbody>
</table>

where \( K \) is determined as in (19.9) and \( L \) is determined as in (19.23). In block matrix form (noting that \( \hat{x} = x - \hat{x} \)), this composite system may be written

\[
\frac{d}{dt} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r + \begin{bmatrix} I & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\]

\( y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \) (19.25a)

Since this system matrix is block triangular, its eigenvalues are given by the union of the eigenvalues of \( A + BK \) and those of \( A + LC \); thus, selecting \( K \) and \( L \) to stabilize the control and estimation problems separately effectively stabilizes the composite system. Further, assuming that \( w_1 = w_2 = 0 \) and the initial condition on all variables are zero, taking the Laplace transform\(^4\) of the composite system (19.25) gives

\[
Y(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sl - (A + BK) & BK \\ 0 & sl - (A + LC) \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} R(s) = C[sl - (A + BK)]^{-1}BR(s).
\]

That is, the transfer function from \( r \) to \( y \) is unaffected by the estimator. Note that, in the SIMO case, this transfer function may be written in the form

\[
\frac{Y(s)}{R(s)} = C[sl - (A + BK)]^{-1}B.
\]

As a matter of practice, the estimator feedback \( L \) is typically designed (by adjusting the relative magnitude of \( Q_1 \) and \( Q_2 \)) such that the slowest eigenvalues of \( A + LC \) are a factor of 2 to 5 faster than the slowest eigenvalues of \( A + BK \).

\(^4\)In effect, simply replacing \( d/dt \) by the Laplace variable \( s \) and replacing the time-domain signals \( \{y,r,x,\hat{x}\} \) with their Laplace transforms \( \{Y(s),R(s),X(s),\hat{X}(s)\} \), where \( F(s) = \int_0^\infty F(t)e^{-st}dt \).