13

Attitude Dynamics

13.1 Aims and Objectives

- To present the universal rotational dynamics model applicable to all aerospace vehicles, emphasizing the commonality between the stability and control characteristics of aircraft and spacecraft.
- To derive several attitude dynamics models based on the useful kinematic parameters introduced in Chapter 2.
- To introduce single-axis, open-loop, time-optimal impulsive maneuvers.
- To present a rigorous derivation of the attitude motion model for atmospheric flight.
- To model and simulate important aerospace attitude motion examples, ranging from spin-stabilized, rotor- and thruster-controlled spacecraft, to gravity gradient satellites, thrust-vectorcd rockets, and six-degree-of-freedom, inertia-coupled, fighter airplanes.

13.2 Euler Equations of Rotational Motion

Up to this point, we have largely confined our attention to the translational motion of flight vehicles, which is represented by the motion of the center of mass. The rotational motion of a vehicle is important for various reasons (aerodynamics, pointing of weapons, payload, or antennas, etc.) and governs the instantaneous attitude (orientation). In Chapter 2, we saw how the attitude of a coordinate frame can be described relative to a reference frame. It was evident that the instantaneous attitude depends not only upon the rotational kinematics, but also on rotational dynamics which determine how the attitude parameters change with time for a specified angular velocity. If we consider a flight vehicle to be rigid, a reference frame attached to the vehicle could be used to represent the vehicle’s attitude. However, in such a case, the angular velocity cannot be an arbitrary parameter but must satisfy the laws of rotational dynamics that take into account the mass distribution of the vehicle. In
this chapter we shall derive the governing equations of rotational dynamics, which are equivalent to Newton’s laws for translational dynamics (Chapter 4).

In Chapter 4, the rotational dynamics of a body—taken to be a collection of particles of elemental mass, $\partial m$—was seen to be described by the following equation of motion derived from Newton’s second law by taking moments about a point $o$, which is either stationary or the body’s center of mass:

$$M = \sum \left( r \times \frac{\partial m}{\partial t} \right). \tag{13.1}$$

Here $v$ is the total (inertial) velocity of the particle, $r$ is the relative position of the particle with respect to the point $o$ (which serves as the origin of a reference coordinate frame), and $M = \sum (r \times \partial f)$ is the net external torque about $o$. In the derivation of Eq. (13.1), it has been assumed that all internal torques cancel each other by virtue of Newton’s third law. This is due to the fact that the internal forces between any two particles constituting the body act along the line joining the particles.\(^1\) By taking the limit $\partial m \to 0$, we can replace the summation over particles by an integral over mass, and write

$$M = \int \left( r \times \frac{dv}{dt} \right) \, dm. \tag{13.2}$$

In Chapter 4, we also defined a particle’s angular momentum by $\partial H = r \times \partial m v$. By integration, the total angular momentum of the body can be written as follows:

$$H = \int r \times v \, dm. \tag{13.3}$$

Assuming that the body has a constant mass, let us differentiate Eq. (13.3) with time, leading to

$$\frac{dH}{dt} = \int \nu \times \nu \, dm + \int \left( r \times \frac{dv}{dt} \right) \, dm. \tag{13.4}$$

The first term on the right-hand side of Eq. (13.6) is identically zero, while the second term is easily identified from Eq. (13.2) to be the net external torque, $M$; thus, we have

$$\frac{dH}{dt} = M. \tag{13.5}$$

Note that in the above derivation, $o$ is either a stationary point or the body’s center of mass. When applied to the general motion of a flight vehicle, it is useful to select $o$ to be the center of mass. In such a case, the moving reference frame, $(oxyz)$, is called a body frame.

Now, let us assume that the body is rigid, i.e., the distance between any two points on the body does not change with time. The rigid-body

\(^1\) Most forces of interaction among particles obey this principle, with the exception of the magnetic force.
assumption—valuable in simplifying the equations of motion—is a reasonable approximation for the rotational dynamics of most flight vehicles. Then it follows that since the center of mass is a point fixed relative to the body (although it may not always lie on the body), the magnitude of the vector \( \mathbf{r} \) is invariant with time. Hence, we can write the total (inertial) velocity of an arbitrary point on the rigid body located at \( \mathbf{r} \) relative to \( o \) as follows:

\[
\mathbf{v} = \mathbf{v}_0 + \mathbf{\omega} \times \mathbf{r},
\]

(13.6)

where \( \mathbf{v}_0 \) denotes the velocity of the center of mass, \( o \), and \( \mathbf{\omega} \) is the angular velocity of the reference coordinate frame with the origin at \( o \). Therefore, for a rigid body, Eqs. (13.3) and (13.6) lead to the following expression for the angular momentum:

\[
\mathbf{H} = \int \mathbf{r} \times \mathbf{v}_0 \, dm + \int \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}) \, dm.
\]

(13.7)

The first term on the right-hand side of Eq. (13.7) can be expressed as

\[
\int \mathbf{r} \times \mathbf{v}_0 \, dm = \left( \int \mathbf{r} \, dm \right) \times \mathbf{v}_0,
\]

(13.8)

which vanishes by the virtue of \( o \) being the center of mass (\( \int \mathbf{r} \, dm = 0 \)). Thus, we have

\[
\mathbf{H} = \int \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}) \, dm.
\]

(13.9)

We choose to resolve all the vectors in the body frame with axes \( ox, oy, oz \) along unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \), respectively, such that

\[
\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k},
\]

(13.10)

\[
\mathbf{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k},
\]

(13.11)

\[
\mathbf{H} = H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k},
\]

(13.12)

\[
\mathbf{M} = M_x \mathbf{i} + M_y \mathbf{j} + M_z \mathbf{k}.
\]

(13.13)

By substituting the vector components into Eq. (13.9) and simplifying, we have the following matrix-vector product for the angular momentum:

\[
\mathbf{H} = \mathbf{J} \mathbf{\omega},
\]

(13.14)

where \( \mathbf{J} \) is the inertia tensor, given by

\[
\mathbf{J} = \begin{pmatrix}
\int (y^2 + z^2) \, dm & -\int xy \, dm & -\int xz \, dm \\
-\int xy \, dm & \int (x^2 + z^2) \, dm & -\int yz \, dm \\
-\int xz \, dm & -\int yz \, dm & \int (x^2 + y^2) \, dm
\end{pmatrix}.
\]

(13.15)

Clearly, \( \mathbf{J} \) is a symmetric matrix. In terms of its components, \( \mathbf{J} \) is written as follows:
The components of the inertia tensor are divided into the moments of inertia, $J_{xx}, J_{yy}, J_{zz}$, and the products of inertia, $J_{xy}, J_{yz}, J_{xz}$. Recall that $\omega$ is the angular velocity of the reference coordinate frame, $(oxyz)$. This frame has its origin, $o$, fixed at the center of mass of the rigid body. However, if the axes of the frame are not fixed to the rigid body, the angular velocity of the body would be different from $\omega$. In such a case, the moments and products of inertia would be time-varying. Since the main advantage of writing the angular momentum in the form of Eq. (13.14) lies in the introduction of an inertia tensor, whose elements describe the constant mass distribution of the rigid body, we want to have a constant inertia tensor. If we deliberately choose to have the axes of the body frame, $(oxyz)$, fixed to the body, and thus rotating with the same angular velocity, $\omega$, as that of the body, the moments and products of inertia will be invariant with time. Such a reference frame, with axes tied rigidly to the body, is called a body-fixed frame. From this point forward, the body frame $(oxyz)$ will be taken to be the body-fixed frame. Hence, $\omega$ in Eq. (13.14) is the angular velocity of the rigid body, and $J$ is a constant matrix.

The equations of rotational motion of the rigid body can be obtained in the body-fixed frame by substituting Eq. (13.14) into Eq. (13.5) and applying the rule of taking the time derivative of a vector (Chapter 2):

$$M = J \frac{\partial \omega}{\partial t} + \omega \times (J \omega),$$

(13.17)

where the partial time derivative represents the time derivative taken with reference to the body-fixed frame,

$$\frac{\partial \omega}{\partial t} = \begin{bmatrix} \frac{d\omega_x}{dt} \\ \frac{d\omega_y}{dt} \\ \frac{d\omega_z}{dt} \end{bmatrix} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix}.$$

(13.18)

By replacing the vector product in Eq. (13.17) by a matrix product (Chapter 2), we can write

$$M = J \frac{\partial \omega}{\partial t} + S(\omega)J\omega,$$

(13.19)

where

$$S(\omega) = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}.$$

(13.20)

Equation (13.19) represents three scalar, coupled, nonlinear, ordinary differential equations, called Euler’s equations of rotational dynamics. These are the governing equations for rotational dynamics of rigid bodies, and their solution
gives the angular velocity, $\omega$, at a given instant. In Chapter 2, we derived the kinematic equations for the rotation of a coordinate frame in terms of various alternative attitude representations. These kinematic equations, along with Euler’s equations of rotational dynamics, complete the set of differential equations needed to describe the changing attitude of a rigid body under the influence of a time-varying torque vector, $\mathbf{M}$. The variables of the rotational motion are thus the kinematical parameters representing the instantaneous attitude of a body-fixed frame, and the angular velocity of the rigid body resolved in the same frame.

13.3 Rotational Kinetic Energy

In Chapter 4, we derived the kinetic energy for a system of $N$ particles

$$T = \frac{1}{2}m v_0^2 + \frac{1}{2} \sum_{i=1}^{N} m_i u_i^2,$$

where $u_i$ is the speed of the $i$th particle (of mass $m_i$) relative to the center of mass $o$, which is moving with a speed $v_0$. When applied to a body, the summation over particles is replaced by an integral over mass, and we have

$$T = \frac{1}{2}m v_0^2 + \frac{1}{2} \int u^2 \, dm.$$

(13.22)

It is clear from Eq. (13.6) that for a rigid body, $u^2 = (\mathbf{\omega} \times \mathbf{r}) \cdot (\mathbf{\omega} \times \mathbf{r})$, and we can write

$$T = \frac{1}{2}m v_0^2 + \frac{1}{2} \int (\mathbf{\omega} \times \mathbf{r}) \cdot (\mathbf{\omega} \times \mathbf{r}) \, dm.$$

(13.23)

The same result could be obtained by using the following defining expression of the kinetic energy, and substituting Eq. (13.6) for a rigid body:

$$T = \frac{1}{2} \int \mathbf{v} \cdot \mathbf{v} \, dm.$$

(13.24)

The first term on the right-hand side of Eq. (13.23) represents the kinetic energy due to the translation of the center of mass, whereas the second term denotes the kinetic energy of rotation about the center of mass. The expression for the rotational kinetic energy of the rigid body, $T_{rot}$, can be simplified by utilizing the angular momentum [Eq. (13.9)], leading to

$$T_{rot} = \frac{1}{2} \int (\mathbf{\omega} \times \mathbf{r}) \cdot (\mathbf{\omega} \times \mathbf{r}) \, dm = \frac{1}{2} \mathbf{\omega} \cdot \mathbf{H} = \frac{1}{2} \mathbf{\omega}^T \mathbf{J} \mathbf{\omega}.$$

(13.25)

This expression for the rotational kinetic energy of a rigid body is very useful in simplifying Euler’s equations.
The rotational kinetic energy is conserved if there is no external torque applied to the rigid body. This fact is evident by taking the time derivative of Eq. (13.25), and substituting Euler’s equations, Eq. (13.18), with \( M = 0 \):

\[
\frac{dT_{\text{rot}}}{dt} = \frac{1}{2} \frac{d\omega}{dt} \cdot H + \frac{1}{2} \omega \cdot \frac{dH}{dt} = 0 .
\]  

(13.26)

Since \( M = 0 \), the second term on the right-hand side of Eq. (13.26) vanishes due to Eq. (13.5), while the first term vanishes by the virtue of Eq. (13.18), which produces \( \frac{d\omega}{dt} \cdot H = -\omega \cdot (\omega \times H) = 0 \). In the absence of an external torque (such as in spacecraft applications), the conservation of both rotational kinetic energy and angular momentum can be effectively utilized in obtaining analytical relationships between the angular velocity and the inertia tensor.

### 13.4 Principal Body Frame

As seen in Chapter 2, the translation of a frame is trivially handled by merely shifting the origin of the coordinate frame. In terms of the body-fixed frame, such a translation of the origin (center of mass of rigid body) would produce a modification of the inertia tensor easily obtained by the parallel axes theorem (discussed later in this chapter). However, a rotation of the body-fixed frame about the same origin is nontrivial. The body-fixed frame used above in deriving Euler’s equations has an arbitrary orientation relative to the rigid body. There are infinitely many ways in which these axes can be fixed to a given rigid body at the center of mass. A great simplification in Euler’s equations is possible by choosing a particular orientation of the body-fixed frame relative to the rigid body such that the products of inertia, \( J_{xy}, J_{yz}, J_{xz} \), vanish. Such a frame is called the principal body-fixed frame. The inertia tensor resolved in the principal body frame is a diagonal matrix, \( J_p \). In order to derive the coordinate transformation that produces the principal frame, \( i_p, j_p, k_p \), from an arbitrary body-fixed frame, \( i, j, k \), consider the rotation matrix, \( C_p \), defined by

\[
\begin{bmatrix}
  i \\
  j \\
  k
\end{bmatrix} = C_p \begin{bmatrix}
  i_p \\
  j_p \\
  k_p
\end{bmatrix} .
\]  

(13.27)

The relationship between a vector resolved in the principal frame and the same vector in an arbitrary body-fixed frame is thus through the rotation matrix, \( C_p \). If we continue to denote the vectors resolved in the principal frame by the subscript \( p \), we have

\[
\omega = C_p \omega_p .
\]  

(13.28)

Now, since there is no change in the rotational kinetic energy caused by the coordinate transformation, we can utilize Eq. (13.25) and write

\[
T_{\text{rot}} = \frac{1}{2} \omega^T J \omega = \frac{1}{2} \omega_p^T J_p \omega_p .
\]  

(13.29)
Upon substituting Eq. (13.28) into Eq. (13.29), and comparing the terms on both the sides of the resulting equation, we have

$$\omega_p^T J_p \omega_p = \omega^T J \omega = \omega_p^T C_p^T J C_p \omega_p,$$  \hspace{1cm} (13.30)

which, on applying the orthogonality property of the rotation matrix, yields

$$J_p = C_p^T J C_p.$$ \hspace{1cm} (13.31)

Since $J_p$ is a diagonal matrix, it easily follows [4] that the diagonal elements of $J_p$ are the distinct eigenvalues of $J$, while $C_p$ has the eigenvectors of $J$ as its columns. Thus, Eq. (13.31) is the formula for deriving the inertia tensor in the principal frame and the coordinate transformation matrix, $C_p$, from the eigenvalue analysis of $J$.

**Example 13.1.** A rigid body has the following inertia tensor:

$$J = \begin{pmatrix} 100 & 10 & 35 \\ 10 & 250 & 50 \\ 35 & 50 & 300 \end{pmatrix} \text{kg.m}^2.$$  

Find the inertia tensor in the principal frame and the coordinate transformation matrix, $C_p$.

This problem is easily solved with the following MATLAB statements employing the intrinsic eigenvalue analysis function *eig.m*:

```matlab
>> J=[100 10 35;10 250 50;35 50 300]; %inertia tensor
>> [Cp,Jp]=eig(J) \ %rotation matrix & principal inertia tensor
Cp =
 0.9862  -0.0754     0.1473
-0.0103     0.8605    0.5094
-0.1651     0.5039    0.8479

Jp =
 94.0366     0         0
 0  219.8462     0
 0         0   336.1172

>> Cp'*J*Cp \ %check the rotation matrix
ans =
 94.0366     0         0
 0  219.8462     0
 0         0   336.1172
```

Thus, the principal inertia tensor is

$$J_p = \begin{pmatrix} 94.0366 & 0 & 0 \\ 0 & 219.8462 & 0 \\ 0 & 0 & 336.1172 \end{pmatrix} \text{kg.m}^2.$$  

The computed rotation matrix represents the orientation of the currently employed body-fixed frame with respect to the principal frame.
The inertia tensor can be diagonalized by the foregoing procedure to produce the principal inertia tensor if and only if the principal moments of inertia are distinct, which is the case for an asymmetric object. For an axisymmetric body, two principal moments of inertia are equal, but it is neither necessary, nor feasible, to follow the above approach for obtaining the principal moments of inertia (since the principal axes are easily identified from symmetry). Hence, for all practical purposes we shall work only in the principal body frame, and the following discussion pertains to the principal body axes, without explicitly carrying the subscript $p$.

### 13.5 Torque-Free Rotation of Spacecraft

A spacecraft’s rotational motion is generally in the absence of external torques. In order to analyze the rotational stability and control characteristics of spacecraft, it thus becomes necessary to study the torque-free motion of rigid bodies. Since the external torque is zero, the angular momentum of the rigid body about its center of mass (or a fixed point) is conserved by the virtue of Eq. (13.5). Thus, we can express Euler’s equations for the torque-free motion ($\mathbf{M} = 0$) of a rigid body in the principal frame as follows:

\begin{align*}
J_{xx}\dot{\omega}_x + \omega_y\omega_z(J_{zz} - J_{yy}) &= 0, \\
J_{yy}\dot{\omega}_y + \omega_x\omega_z(J_{xx} - J_{zz}) &= 0, \\
J_{zz}\dot{\omega}_z + \omega_x\omega_y(J_{yy} - J_{xx}) &= 0, \tag{13.32}
\end{align*}

where the dot represents the time derivative, $\frac{d}{dt}$. For a general, asymmetric body possessing nonzero angular velocity components about all three axes, Eq. (13.32) is difficult to solve in a closed form, but is amenable to numerical integration in time.

Since a torque-free rigid body does not have a mechanism for energy dissipation, its rotational kinetic energy is conserved according to Eq. (13.26). However, a spacecraft is an imperfect rigid body, generally consisting of several rigid bodies rotating relative to each other (e.g., reaction wheels and control gyroscopes), as well as containing liquid propellants. The rotors and liquid propellants provide mechanisms for internal dissipation of the rotational kinetic energy through friction and sloshing motion, respectively. When analyzing the rotational stability of spacecraft, it is therefore vital to regard them as semirigid objects that continually dissipate kinetic energy until a stable equilibrium is achieved. For a semirigid body, Euler’s equations remain valid (as the external torque remains zero), but the rotational kinetic energy is not conserved.

Before solving torque-free Euler equations for a general case, let us use them to analyze rotational stability characteristics of rigid spacecraft. Such an analysis would reveal the axes about which a stable rotational equilibrium can be achieved. The process of obtaining a stable equilibrium through
constant speed rotation about a principal axis is called spin stabilization. Although spin stabilization is strictly valid only for a spacecraft, it can be applied approximately to some atmospheric flight vehicles that have a small aerodynamic moment about the spin axis, such as certain missiles and projectiles. A rifle bullet is a good example of a spin-stabilized object. Furthermore, spin stabilization is also the principle of operation of gyroscopic instruments, which are commonly used in aerospace vehicles.

Stability is a property of an equilibrium and can be defined in many ways. For our purposes, we shall define a stable equilibrium as the one about which a bounded disturbance does not produce an unbounded response. The disturbance can be regarded as the initial condition, expressed in terms of an initial deviation of the motion variables from the equilibrium. In a stability analysis, it is sufficient to study the response to a small initial deviation, because stability is not influenced by the magnitude of the disturbance.

13.5.1 Axisymmetric Spacecraft

When the spacecraft possesses an axis of symmetry, Euler’s equations are further simplified. Consider a spacecraft rotating about its axis of symmetry, oz, called the longitudinal axis. Due to axial symmetry, \( J_{xx} = J_{yy} \), and we have

\[
\begin{align*}
J_{xx} \dot{\omega}_x + \omega_y \omega_z (J_{zz} - J_{xx}) &= 0, \\
J_{xx} \dot{\omega}_y + \omega_x \omega_z (J_{xx} - J_{zz}) &= 0, \\
J_{zz} \dot{\omega}_z &= 0.
\end{align*}
\] (13.33)

It is clear from Eq. (13.33) that the spacecraft is in a state of equilibrium whenever \( \omega_x = \omega_y = 0 \), called pure spin about the axis of symmetry. It is also evident from the last of Eq. (13.33) that \( \dot{\omega}_z = 0 \), or \( \omega_z = n = \text{constant} \), irrespective of the magnitudes of \( \omega_x, \omega_y \). Let us assume that the spacecraft was in a state of pure spin when a disturbance, \( \omega_x(0), \omega_y(0) \), is applied at time \( t = 0 \). Let us examine the resulting motion of the spacecraft by solving the first two equations of Eq. (13.33), which are written in the following vector matrix form:

\[
\begin{pmatrix}
\omega_x \\
\omega_y
\end{pmatrix}
= \begin{pmatrix}
0 & -k \\
k & 0
\end{pmatrix}
\begin{pmatrix}
\omega_x \\
\omega_y
\end{pmatrix},
\] (13.34)

where \( k = n (J_{xx} - J_{yy}) \). Equation (13.34) represents linear, time-invariant state equations (Chapter 14) whose solution with the initial condition, \( \omega_x(0), \omega_y(0) \) at \( t = 0 \), is easily written in a closed form as follows:

\[
\begin{pmatrix}
\omega_x(t) \\
\omega_y(t)
\end{pmatrix}
= e^{kt}
\begin{pmatrix}
\omega_x(0) \\
\omega_y(0)
\end{pmatrix},
\] (13.35)
where $e^{Kt}$ is the matrix exponential (Chapter 14), and

$$K = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}. \quad (13.36)$$

Using one of the methods of Chapter 14, we can write the matrix exponential by taking the inverse Laplace transform of the resolvent as follows:

$$e^{Kt} = \mathcal{L}^{-1}(sI - K)^{-1} = \begin{pmatrix} \cos(kt) & -\sin(kt) \\ \sin(kt) & \cos(kt) \end{pmatrix}. \quad (13.37)$$

Therefore, the solution is given by

$$\begin{align*}
\omega_x(t) &= \omega_x(0) \cos(kt) - \omega_y(0) \sin(kt), \\
\omega_y(t) &= \omega_x(0) \sin(kt) + \omega_y(0) \cos(kt). \quad (13.38)
\end{align*}$$

Equation (13.33) implies that the rotational motion of an axisymmetric, rigid spacecraft, disturbed from the equilibrium state of pure spin about the longitudinal axis by a disturbance $\omega_x(0), \omega_y(0)$, is oscillatory in the $oxy$ plane (called the *lateral plane*), while the spin rate, $\omega_z = n$, remains unaffected. This causes a coning motion of the disturbed body about the axis of symmetry. An important characteristic of the solution given by Eq. (13.38) is easily seen to be the following:

$$\omega_{xy}^2 = \omega_x^2 + \omega_y^2 = \omega_x^2(0) + \omega_y^2(0) = \text{constant}, \quad (13.39)$$

which implies that the magnitude of the angular velocity component in the lateral plane is constant. This lateral angular velocity component, $\omega_{xy}$, is responsible for the coning motion called *precession*. Since precession is a constant amplitude oscillation, whose magnitude is bounded by that of the applied disturbance, we say that the motion of a rigid spacecraft about its axis of symmetry is unconditionally stable. Figure 13.1 shows the geometry of precessional motion, where the angular velocity, $\omega$, makes a constant angle, $\alpha = \tan^{-1} \frac{\omega_x}{\omega_y}$, with the axis of symmetry, oz. Furthermore, the angular momentum, $\mathbf{H} = J_{xx}(\omega_x \mathbf{i} + \omega_y \mathbf{j}) + J_{zz} \mathbf{n} \mathbf{k}$, makes a constant angle, $\beta = \tan^{-1} \frac{J_{xx} \omega_x}{J_{zz} \omega_y}$, with the axis of symmetry, called the *nutation angle*. The axis of symmetry thus describes a cone of semivertex angle $\alpha$, called the *body cone*, about the angular velocity vector, and a cone of semivertex angle $\beta$, called the *space cone*, about the angular momentum vector. Note that while $\omega$ is a rotating vector, $\mathbf{H}$ is fixed in inertial space due to the conservation of angular momentum. In Fig. 13.1, $J_{xx} > J_{zz}$ is assumed, for which $\beta > \alpha$. While a rigid, axisymmetric spacecraft’s precessional motion is unconditionally stable (as seen above), the same cannot be said for a semirigid spacecraft. Since most spacecraft carry some liquid propellants, they must be regarded as semirigid, wherein the angular momentum is conserved, but the rotational kinetic energy dissipates due to the sloshing of liquids caused by precession.
Whenever energy dissipation is present in a dynamical system, there is a tendency to move toward the state of equilibrium with the lowest kinetic energy. In a state of pure spin about a principal axis, there is no energy dissipation because the liquids rotate with the same speed as the spacecraft. Thus, pure spin about a principal axis spin is a state of equilibrium for a semirigid spacecraft. For the torque-free rotation of spacecraft, the lowest kinetic energy is achieved for pure spin about the major principal axis. This can be seen from Eq. (13.25), while applying the law of conservation of angular momentum. Hence, the internal energy dissipation eventually converts the precessional motion into a spin about the major axis. Therefore, a semirigid spacecraft can be spin-stabilized only about its major axis. In applying the foregoing results to such a spacecraft, it is necessary that $J_{zz} > J_{xx}$. If the axis of symmetry is the minor axis, pure spin about it would eventually be converted into a tumbling motion about the major principal axis in the presence of inevitable disturbances and liquid propellants. This phenomenon was encountered in the first satellite launched by NASA, named Explorer, rendering the long cylindrical spacecraft useless after a few days in orbit. For this reason, all spinning satellites are designed to have the axis of symmetry as the major axis. One may study the attitudinal kinematics of axisymmetric spacecraft, spin stabilized about the longitudinal axis, by simultaneously solving the kinematic equations of motion (Chapter 2) with the Euler equations. Since the angular momentum vector, $H$, is fixed in space, an obvious choice of the reference inertial frame is with the axis $K$ along $H$. The most commonly used kinematic parameters for spin-stabilized spacecraft are the $(\psi)_3, (\theta)_1, (\phi)_3$ Euler angles (Chapter 2). Since the spin axis of the precessing spacecraft is never exactly aligned with the angular momentum ($\theta \neq 0$), the singularity of this attitude...
Fig. 13.2. Attitude of a precessing, axisymmetric spacecraft via 3-1-3 Euler angles.

representation at $\theta = 0, 180^\circ$ is not encountered, thereby removing the main disadvantage of Euler’s angle representation. Therefore, the constant nutation angle is given by $\beta = \theta$, and from Fig. 13.2 depicting Euler’s angles, we have

$$
\sin \theta = \frac{J_{xx}\omega_{xy}}{H} = \frac{J_{xx}\omega_{xy}}{\sqrt{J_{xx}\omega_{xy}^2 + J_{zz}n^2}}
$$

$$
\cos \theta = \frac{J_{zz}n}{H} = \frac{J_{zz}n}{\sqrt{J_{xx}\omega_{xy}^2 + J_{zz}n^2}}
$$

(13.40)

The general kinematic equations for the $(\psi)_3, (\theta)_1, (\phi)_3$ Euler angles were derived in Chapter 2 and are repeated here as follows:

$$
\begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} = \frac{1}{\sin \theta} \begin{bmatrix}
\sin \phi & \cos \phi & 0 \\
\cos \phi \sin \theta & -\sin \phi \sin \theta & 0 \\
-\sin \phi \cos \theta & -\cos \phi \cos \theta & \sin \theta
\end{bmatrix} \begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}.
$$

(13.41)

Upon substitution of $\omega_z = n$, and Eq. (13.39) into Eq. (13.41) we have

$$
\dot{\psi} = \frac{\omega_{xy}}{\sin \theta},
$$

$$
\dot{\theta} = 0,
$$

$$
\dot{\phi} = n - \frac{\omega_{xy}}{\tan \theta}.
$$

(13.42)

Since both $\theta$ and $\omega_{xy}$ are constants [Eqs. (13.39) and (13.40)], the angular rates $\dot{\psi}$ and $\dot{\phi}$ are also constants, whose alternative expressions are obtained by substituting Eq. (13.40) into Eq. (13.42) as
\[ \dot{\psi} = \sqrt{\frac{J_{xx} \omega^2_{xy} + J_{zz} n^2}{J_{xx}}}, \]
\[ \dot{\theta} = 0, \]
\[ \dot{\phi} = n(1 - \frac{J_{zz}}{J_{xx}}) = -k. \] (13.43)

The angular rate \( \dot{\psi} \) represents the frequency of precession and is called the
precession rate, while \( \dot{\phi} \) represents the total spin rate of the body in the inertial
frame and is known as the inertial spin rate. If \( J_{xx} > J_{zz} \), the axisymmetric
body is said to be prolate, and \( \dot{\psi} \) has the same sign as that of \( \dot{\phi} \). For the
case of an oblate body \( (J_{xx} < J_{zz}) \), the angular rates \( \dot{\psi} \) and \( \dot{\phi} \) have opposite
signs. The solution for the Euler angles is easily obtained by integration of
Eq. (13.43)—with the initial orientation at \( t = 0 \) specified as \( \psi(0), \theta(0), \phi(0) \)—
to be the following:
\[ \psi = \psi_0 + \sqrt{\frac{J_{xx} \omega^2_{xy} + J_{zz} n^2}{J_{xx}}} t, \]
\[ \theta = \theta(0), \]
\[ \phi = \phi(0) - nt = \phi(0) - n(1 - \frac{J_{zz}}{J_{xx}}) t. \] (13.44)

The angles \( \psi \) and \( \phi \) thus vary linearly with time due to a constant precession
rate, \( \omega_{xy} \).

### 13.5.2 Asymmetric Spacecraft

Let us assume that an asymmetric spacecraft is in a state of pure spin of
rate \( n \) about the principal axis \( oz \), prior to the time \( t = 0 \) when a small
disturbance, \( \omega_x(0), \omega_y(0) \), is applied. At a subsequent time, the angular velocity
components can be expressed as \( \omega_z = n + \epsilon \), and \( \omega_x, \omega_y \). Since a small distur-
bance has been applied, we can treat \( \epsilon, \omega_x, \omega_y \) as small quantities and solve
Euler’s equations. If the solution indicates that \( \epsilon, \omega_x, \omega_y \) grow with time in an
unbounded fashion, it will be evident that our assumption of small deviations
remaining small is false, and we are dealing with an unstable equilibrium.
Otherwise, we have a stable equilibrium. Hence, with the assumption of small
deviation from equilibrium, we can write the approximate, linearized Euler
equations as follows:
\[ J_{xx} \dot{\omega}_x + n \omega_y (J_{zz} - J_{yy}) \approx 0, \]
\[ J_{yy} \dot{\omega}_y + n \omega_x (J_{xx} - J_{zz}) \approx 0, \]
\[ J_{zz} \epsilon \approx 0, \] (13.45)
in which we have neglected second- (and higher-) order terms involving
\( \epsilon, \omega_x, \omega_y \). The first two equations of Eq. (13.45) can be written in the fol-
lowing vector matrix form:
\[
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y
\end{bmatrix} = 
\begin{pmatrix}
0 & -k_1 \\
k_2 & 0
\end{pmatrix} 
\begin{bmatrix}
\omega_x \\
\omega_y
\end{bmatrix},
\]

(13.46)

where \(k_1 = n \frac{(J_{zz} - J_{yy})}{J_{xx}}\) and \(k_2 = n \frac{(J_{zz} - J_{xx})}{J_{yy}}\). Being in a linear, time-invariant state-space form, these approximate equations are solved using the matrix exponential as follows:

\[
\begin{bmatrix}
\omega_x(t) \\
\omega_y(t)
\end{bmatrix} = e^{At} \begin{bmatrix}
\omega_x(0) \\
\omega_y(0)
\end{bmatrix},
\]

(13.47)

where \(e^{At}\) is the matrix exponential denoting the state transition matrix (Chapter 14) and

\[
A = \begin{pmatrix}
0 & -k_1 \\
k_2 & 0
\end{pmatrix}.
\]

(13.48)

The eigenvalues of \(A\) determine whether the ensuing motion will be bounded, and thus denote stability or instability. They are obtained as follows:

\[
|sI - K| = s^2 + k_1k_2 = 0,
\]

(13.49)

or,

\[
s_{1,2} = \pm \sqrt{-k_1k_2}.
\]

(13.50)

From the eigenvalues of \(A\), it is clear that two possibilities exist for the response: (a) \(k_1k_2 < 0\), for which one eigenvalue has a positive real part, indicating exponentially growing (unbounded) motion, or (b) \(k_1k_2 > 0\), for which both eigenvalues are imaginary, and the motion is a constant amplitude (bounded) oscillation about the equilibrium. Therefore, for stability we must have \(k_1k_2 > 0\), which implies that either \((J_{zz} > J_{xx}, J_{zz} > J_{yy})\) or \((J_{zz} < J_{xx}, J_{zz} < J_{yy})\). Hence, spin stabilization of a rigid, asymmetric spacecraft is possible about either the major principal axis or the minor principal axis. This confirms our conclusion of the previous section, where the axisymmetric spacecraft (which, by definition, has only major and minor axes) was seen to be unconditionally stable. However, if we take into account the internal energy dissipation, the analysis of the previous section dictates that an asymmetric, semirigid spacecraft can be spin-stabilized only about the major axis.

There is a major difference in the stable oscillation of the asymmetric spacecraft from that of the axisymmetric spacecraft studied in the previous section. Due to the presence of a nonzero, bounded disturbance, \(\epsilon\), about the spin axis, the angular velocity component, \(\omega_z = n + \epsilon\), does not remain constant in the case of the asymmetric body. This translates into a nodding motion of the spin axis, wherein the nutation angle, \(\beta\), changes with time. Such a motion is called nutation of the spin axis and is superimposed on the precessional motion.²

² Certain textbooks and research articles on space dynamics use nutation interchangeably with precession, which is incorrect and causes untold confusion. The
Assuming $k_1 k_2 > 0$, we have from Eq. (13.47),

$$e^{k_1} = L^{-1}(sI - K)^{-1} = \begin{pmatrix} \cos(\sqrt{k_1 k_2} t) & -\sqrt{\frac{k_1}{k_2}} \sin(\sqrt{k_1 k_2} t) \\ \sqrt{\frac{k_1}{k_2}} \sin(\sqrt{k_1 k_2} t) & \cos(\sqrt{k_1 k_2} t) \end{pmatrix}. \quad (13.51)$$

Therefore, the approximate, linearized solution for precessional motion for small disturbance is given by

$$\omega_x(t) = \omega_x(0) \cos(\sqrt{k_1 k_2} t) - \omega_y(0) \sqrt{\frac{k_1}{k_2}} \sin(\sqrt{k_1 k_2} t),$$

$$\omega_y(t) = \omega_x(0) \sqrt{\frac{k_2}{k_1}} \sin(\sqrt{k_1 k_2} t) + \omega_y(0) \cos(\sqrt{k_1 k_2} t). \quad (13.52)$$

In order to solve for the nutation angle, we must integrate the last equation of Eq. (13.32). However, by consistently neglecting the second-order term, $\omega_x \omega_y$, in this equation due to the assumption of small disturbance, we have obtained an erroneous result of $\dot{\epsilon} = 0$, or $\epsilon = \text{constant}$ in Eq. (13.45). Hence, the linearized analysis is insufficient to model the nutation of an asymmetric body. We must drop the assumption of small disturbance and numerically integrate the complete, torque-free, nonlinear Euler equations, Eq. (13.32), for an accurate simulation of the combined precession and nutation.

The kinematic equations for the instantaneous attitude of the asymmetric spacecraft in terms of the Euler angles are given by Eq. (13.41), with the choice (as before) of the constant angular momentum vector as the $K$-axis of the inertial frame. By a simultaneous, numerical integration of the kinematic equations along with the nonlinear Euler’s equations, we can obtain the instantaneous attitude of the asymmetric, rigid spacecraft.

**Example 13.2.** A rigid spacecraft with principal moments of inertia $J_{xx} = 4000 \, \text{kg} \cdot \text{m}^2$, $J_{yy} = 7500 \, \text{kg} \cdot \text{m}^2$, and $J_{zz} = 8500 \, \text{kg} \cdot \text{m}^2$ has initial angular velocity $\omega(0) = (0.1, -0.2, 0.5)^T$ rad/s and an initial attitude $\psi(0) = 0, \theta(0) = \frac{\pi}{2}, \phi(0) = 0$. Simulate the subsequent rotation of the spacecraft.

Since the given initial condition is relatively large, the approximation of small disturbance is invalid, and both precession and nutation must be properly simulated. We carry out the three-degree-of-freedom simulation by solving the nonlinear, torque-free Euler equations, and the kinematic equations.

dictionary in this regard is very helpful: “precession” is derived from the Latin word *praecedere*, which means the *act of preceding* and is directly relevant to the motion of a spinning, axisymmetric, prolate body, wherein the rotation of spin axis, $\dot{\psi}$, precedes the spinning motion, $\omega_z = n$, itself. On the other hand, “nutation” is derived from the Latin word *nutare*, which means to nod and describes the nodding motion, $\dot{\theta}$, of the spin axis.

3 Jacobi [41] derived a closed-form solution for Euler's equations of torque-free, asymmetric spacecraft [Eq. (13.32)] in terms of the Jacobian elliptic functions. However, due to the complexity in evaluating these functions [2], we shall avoid their use here and carry out numerical integration of Eq. (13.32).
Eq. (13.41), with the use of a fourth-order Runge-Kutta algorithm (Appendix A) encoded in the intrinsic MATLAB function, ode45.m. The time derivatives of the motion variables, $\omega_x, \omega_y, \omega_z, \psi, \theta, \phi$, required by ode45.m are supplied by the program spacerotation.m, which is tabulated in Table 13.1. The simulation is carried out for 40 s by specifying the initial condition in the call for ode45.m as follows:

```matlab
>> [t,x]=ode45(@spacerotation,[0 40],[0.1 -0.2 0.5 0 0.5*pi 0]);
>> subplot(121),plot(t,x(:,1:3)*180/pi),hold on,...
    subplot(122),plot(t,x(:,4:6)*180/pi)%time evolution of motion variables
```

**Table 13.1.** M-file spacerotation.m for the Torque-free Equations of Rotational Motion

```matlab
function xdot=spacerotation(t,x)
%program for torque-free rotational dynamics and Euler 3-1-3 kinematics
%x=[omega_x, omega_y, omega_z, psi, theta, phi] (angular velocity in rad/s)
%J1=4000; J2=7500; J3=8500; %principal moments of inertia (kg.m^2)
%xdot(1,1)=x(2)*x(3)*(J2-J3)/J1;
%xdot(2,1)=x(1)*x(3)*(J3-J1)/J2;
%xdot(3,1)=(sin(x(6))*x(1)+cos(x(6))*x(2))/sin(x(5));
%xdot(4,1)=cos(x(6))*x(1)-sin(x(6))*x(2);
%xdot(5,1)=x(3)-(sin(x(6))*cos(x(5))*x(1)+cos(x(6))*cos(x(5))*x(2))/sin(x(5));
```

The resulting time-history plots of the motion variables are shown in Fig. 13.3. The precession is evident in the oscillation of $\omega_x, \omega_y, \psi, \phi$, while the nutation is observed in the the oscillation of $\omega_z, \theta$. Such a complex motion would be completely missed in a simulation with the approximate, linearized equations, Eq. (13.45), whereby an erroneous result of $\omega_z = 0.5$ rad/s ($= 28.65^\circ/s$) would be obtained.

### 13.6 Spacecraft with Attitude Thrusters

Spin stabilization of torque-free spacecraft is a cheap (fuel-free) and simple procedure, compared to stabilization with externally applied torques. However, controlling the motion of a spinning body for carrying out the necessary attitude maneuvers is a complex task. Generally, all spacecraft have a reaction control system (RCS) that employs a pair of rocket thrusters—called attitude thrusters—about each principal axis for performing attitude maneuvers. When torques about each principal axis are applied for stability and control, the spacecraft is said to be three-axis stabilized, as opposed to spin-stabilized.

The attitude thrusters of an RCS are operated in pairs with equal and opposite thrust, such that the net external force remains unaffected. The
firing of thrusters is limited to short bursts, which can be approximated by torque impulses. A torque impulse is defined as a torque of infinite magnitude acting for an infinitesimal duration, thereby causing an instantaneous change in the angular momentum of the spacecraft about the axis of application.

The concept of the torque impulse is very useful in analyzing the single-axis rotation of spacecraft, as it allows us to utilize the well-known linear system theory [43], wherein the governing linear differential equation is solved in a closed form with the use of the unit impulse function, \( \delta(t) \), which denotes an impulse of unit magnitude.\(^4\) The change in angular momentum caused by an impulsive torque, \( M(t) = M(0)\delta(t) \), can be obtained as the total area under the torque vs. time graph, given by

\[
\Delta H = \int_{-\infty}^{\infty} M(t) dt = \int_{-\infty}^{\infty} M(0)\delta(t) dt = M(0) .
\]  

\(^4\) The unit impulse function (also known as the Dirac delta function), \( \delta(t - t_0) \), denoting a unit impulse applied at time \( t = t_0 \), has the useful property,

\[
\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0),
\]

where \( f(t) \) is a single-valued function.
Thus, the torque impulse causes an instantaneous change in the angular momentum, equal to the value of the torque at the instant of impulse application, \( t = 0 \).

### 13.6.1 Single-Axis Impulsive Rotation

A complex maneuver can be designed as a sequence of single-axis rotations, for which the \textit{time-optimal}, linear control theory \cite{ref42} is most amenable. Consider a rigid spacecraft with moment of inertia, \( J_{zz} \), about the axis of desired rotation, \( oz \), and equipped with a pair of attitude thrusters capable of exerting a large, maximum torque, \( M_z(0) \), for an infinitesimal duration, \( \Delta t \to 0 \), which causes an instantaneous change in the angular momentum by \( \Delta H_z = M_z(0) \).

Since the torque as a function of time is given by \( M_z(t) = M_z(0)\delta(t) \), Euler’s equations reduce to the following:

\[
\begin{align*}
\dot{\omega}_x &= 0, \\
\dot{\omega}_y &= 0, \\
J_{zz}\dot{\omega}_z &= M_z(0)\delta(t).
\end{align*}
\] (13.54)

In terms of the angular displacement about \( oz, \theta \), the last of Eq. (13.54) can be written as

\[
\dot{\theta} = \frac{M_z(0)}{J_{zz}}\delta(t),
\] (13.55)

whose solution is easily obtained by successive integration using Laplace transform \cite{ref4} to be

\[
\begin{align*}
\omega_z(t) &= \dot{\theta} = \omega_z(0) + \frac{M_z(0)}{J_{zz}}u_s(t), \\
\theta(t) &= \theta(0) + \omega_z(0)t + \frac{M_z(0)}{J_{zz}}r(t),
\end{align*}
\] (13.56)

where \( \theta(0), \omega_z(0) \) refer to the initial condition immediately before torque application, \( u_s(t) = \int \delta(t)dt \) is the \textit{unit step function} applied at \( t = 0 \), defined by

\[
u_s(t - t_0) = \begin{cases} 0, & t < t_0, \\ 1, & t \geq t_0, \end{cases}
\] (13.57)

\footnote{Time-optimal control, as the name implies, is a special branch of optimal control theory, which deals with the problem of optimizing time in a general dynamical system. When the applied inputs are limited in magnitude (such as in the case of rocket thrusters), and the system is governed by linear differential equations, the maximum principle of Pontryagin dictates that the inputs of the maximum possible magnitude should be applied in order to minimize the total time of a given displacement of the system. Pontryagin’s principle is directly applicable to single-axis maneuvers of rigid spacecraft by attitude thrusters.}
and \( r(t) = \int u_s(t)\,dt \) is the unit ramp function applied at \( t = 0 \), defined by

\[
    r(t - t_0) = \begin{cases} 
        0, & t < t_0 \\
        t - t_0, & t \geq t_0
    \end{cases}.
\]  

(13.58)

In a practical application, the thruster torque, \( M_z(0) \), is not infinite, and the time interval, \( \Delta t \), over which the torque acts, tends to zero. However, since \( \Delta t \) is much smaller than the period of the maneuver, it is a good approximation (and a valuable one) to assume an impulsive thruster torque, and to employ Eq. (13.56) as the approximate solution. Equation (13.56) implies that the response to a single impulse is a linearly increasing displacement and a step change in the speed. Therefore, if the maneuvering requirement is for a step change in angular velocity (called a spin-up maneuver), a single impulse is sufficient. However, if a given single-axis displacement is desired—called a rest-to-rest maneuver—one has to apply another impulse of opposite direction, \(-M_z(0)\delta(t - \tau)\), in order to stop the rotation at time \( t = \tau \), when the desired displacement has been reached. Since the governing differential equation, Eq. (13.55), is linear, its solution obeys the principle of linear superposition [43], which allows a weighted addition of the responses to individual impulses to yield the total displacement caused by multiple impulses. Therefore, the net response to two equal and opposite impulses applied after an interval \( t = \tau \) is given by

\[
    \omega_z(t) = \frac{M_z(0)}{J_{zz}}[u_s(t) - u_s(t - \tau)],
\]

\[
    \theta(t) = \frac{M_z(0)}{J_{zz}}[r(t) - r(t - \tau)] + \omega_z\tau = \theta_d.
\]  

(13.59)

Hence, the angular velocity becomes zero, and a desired constant displacement, \( \theta(t) = \theta_d \), is reached at \( t = \tau \). The magnitude of \( \theta_d \) can be controlled by varying the time \( \tau \) at which the second impulse is applied (Fig. 13.4). The application of two equal and opposite impulses of maximum magnitude for achieving a time-optimal displacement is called bang-bang control. This is an open-loop control, requiring only the desired displacement, as opposed to closed-loop control [43], for which the knowledge of instantaneous displacement, \( \theta(t) \), is also required. The bang-bang, time-optimal, open-loop control is exactly applicable to any linear system without resistive and dissipative external forces. However, even when a small damping force is present, one can approximately apply this approach to control linear systems.

### 13.6.2 Attitude Maneuvers of Spin-Stabilized Spacecraft

Attitude thrusters can be used for controlling the attitude of a spin-stabilized, axisymmetric spacecraft, which involves multi-axis rotation (precession). If the spin rate is constant (\( \omega_z = n \)), the governing differential equations describing
precession, Eq. (13.42), are linear, thus enabling the use of time-optimal, bang-bang, open-loop control in the same manner as the single-axis rotation. In order to apply the bang-bang approach, the precessional motion is excited by applying a torque normal to the spin axis and then exerting another equal and opposite torque to stop the precession when the desired spin-axis orientation has been reached. However, contrary to single-axis rotation, the principal axes of a precessing body are not fixed in space. Hence, the directions of the two torque impulses are referred to the inertial axes.

Let a change of the spin axis be desired through application of thruster torque impulses, as shown in Fig. 13.5. After the application of the first im-
pulse, $\Delta H_1$, the angular momentum changes instantaneously from $H_0 = J_{zz}n k$ to its new value $H_1 = H_0 + \Delta H_1$, such that a nutation angle of $\beta = \frac{\theta_d}{2}$ is obtained. We select the orientation of the inertial frame such that $oZ$ is along the intermediate angular momentum vector, $H_1$, and $oX$ coincides with the principal axis $ox$ at time $t = 0$. Therefore, we have $\psi(0) = 0, \theta(t) = \frac{\theta_d}{2}, \phi(0) = 0$ in terms of the 3-1-3 Euler angles. It is clear from Fig. 13.5 that the first torque impulse applied normal to the spin axis at $t = 0$ is equal to

$$\Delta H_1 = J_{zz}n \tan \frac{\theta_d}{2} \left( \cos \frac{\theta_d}{2} J + \sin \frac{\theta_d}{2} K \right) = J_{zz}n \tan \frac{\theta_d}{2} j,$$

and causes a positive rotation of the angular momentum vector about $-I$. Since the angular momentum has been deflected from the spin axis, the precessional motion is excited and is allowed to continue for half inertial spin ($\phi = \pi$) until $i = -I$. At that precise instant, the second impulse,

$$\Delta H_2 = J_{zz}n \tan \frac{\theta_d}{2} \left( \cos \frac{\theta_d}{2} J - \sin \frac{\theta_d}{2} K \right) = J_{zz}n \tan \frac{\theta_d}{2} j,$$

is applied in order to stop the precession by causing a positive rotation of the angular momentum vector about $I$. The angular momenta at the beginning and end of the precession are given in terms of the instantaneous principal axes by

$$H_1 = J_{zz}n k + J_{zz}n \tan \frac{\theta_d}{2} j,$$

$$H_2 = J_{zz}n k.$$

It is important to emphasize that the principal axes used in the expressions for $H_1$ and $H_2$ are at different instants, separated in time by half the inertial spin time period. The time taken to undergo half inertial spin is given by Eq. (13.42) to be

$$t_{1/2} = \frac{\pi}{\dot{\phi}} = \frac{J_{xx} \pi}{n |J_{xx} - J_{zz}|}.$$

It is clear from Eq. (13.63) that the time it takes to reach the final position is large if the spin rate, $n$, is small or if the two moments of inertia are close to each other.

Although the two impulses are opposite in direction relative to the inertial frame, they have the same orientation in the the instantaneous body-fixed principal frame. Hence, the same pair of attitude thrusters can be used to both start and stop the precession after multiples of half inertial spin ($\phi = \pm \pi, 2\pi, \ldots$). However, in order to achieve the largest possible deflection of the spin axis—which is equal to $\theta_d$ and happens when $H_0, H_1, H_2$ all lie in the same plane—the precession angle, $\psi$, must have changed exactly by $\pm 180^\circ$ when the precession is stopped, which requires that $|\dot{\psi}| = |\dot{\phi}|$. On equating the magnitudes of the inertial spin and precession rates in Eq. (13.42), it is
clear that the matching of precession with inertial spin is possible if and only if
\[
\cos \frac{\theta_d}{2} = \frac{J_{zz}}{|J_{xx} - J_{zz}|},
\]
(13.64)

Because the cosine of an angle cannot exceed unity, this implies that precession and inertial spin can be synchronized only for prolate bodies with \( J_{xx} > 2J_{zz} \). Equation (13.64) gives the largest possible angular deflection of the spin axis \( \theta_d \) that can be achieved with a given pair of attitude thrusters and is obtained when \( |\psi| = |\phi| = \pi \). Since the nutation angle, \( \beta = \frac{\theta_d}{2} \), is determined purely by the impulse magnitude, its value can be different from that given by Eq. (13.64), in which case the total angular deviation of the spin axis is less than \( 2\beta \). From the foregoing discussion, it is clear that for a greater flexibility in performing spin-axis maneuvers, more than one pair of attitude thrusters (or more than two impulses) should be employed.

Since the applied torque magnitude for each impulse, \( M_y \), is proportional to \( \tan(\frac{\theta_d}{2}) \), it follows that a change of spin axis by \( \theta_d = 180^\circ \) would be infinitely expensive. Because impulsive maneuvers are impossible in practice, one must take into account the nonzero time, \( \Delta t \), of thruster firing, which leads to an average thruster torque requirement \( M_y = \frac{\Delta H_1}{\Delta t} \). In simulating the spacecraft response due to thruster firing, one must carefully model the actual variation of the thruster torque with time. There are two distinct ways of simulating the bang-bang, impulse response of spin-stabilized, axisymmetric spacecraft: (a) calculating the precessional angular velocity components, \( \omega_x, \omega_y \), due to the applied impulses, and using them as an initial condition to simulate the ensuing torque-free motion, or (b) directly simulating the response to the
applied impulses by solving the equations of motion with a nonzero torque. Of these two, the former is an initial response describing precession between the two impulses, while the latter includes the impulse response caused by the impulses themselves. Since \( \theta \neq 0 \) for the first method, we can use the 3-1-3 Euler angles for a nonsingular attitude simulation. However, the second approach begins with a zero nutation angle before the application of the first impulse; thus, the 3-1-3 Euler angle representation is unsuitable; instead, the 3-2-1 Euler angle representation should be employed in (b). The kinematic equations of motion in terms of the \((\psi)_3, (\theta)_2, (\phi)_1\) Euler angles are easily derived using the methods of Chapter 2 to be

\[
\begin{pmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{pmatrix} = \frac{1}{\cos \theta}
\begin{pmatrix}
0 & \sin \phi & \cos \phi \\
0 & \cos \phi \cos \theta - \sin \phi \cos \theta & 0 \\
1 & \sin \phi \sin \theta & \cos \phi \sin \theta
\end{pmatrix}
\begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix}.
\] (13.65)

Here we employ the initial angular momentum vector, \(H_0\), to be the inertial axis, \(\text{oZ}\). Therefore, the nutation angle, \(\beta\), is given by

\[
\cos \beta \equiv K \cdot k = \cos \theta \cos \phi,
\] (13.66)

which determines \(\beta\) uniquely, as \(\beta \leq \pi\). However, in this case, the nutation angle, \(\beta\), denotes the total deviation of the spin axis from its original position (rather than the deviation from the intermediate angular momentum, \(H_1\), of the 3-1-3 Euler angle representation shown in Fig. 13.5). We have seen above that a 180° deflection of the spin axis requires an infinite impulse magnitude, which is practically impossible. Hence, we are necessarily simulating an impulse response with \(\beta < \pi\), for which the 3-2-1 Euler angles are nonsingular. We shall apply both simulation methods, (a) and (b), in the following example.

**Example 13.3.** Consider an axisymmetric, spin-stabilized, rigid spacecraft with principal moments of inertia \(J_{xx} = J_{yy} = 1500 \text{ kg.m}^2\) and \(J_{zz} = 500 \text{ kg.m}^2\) and spin rate \(\omega_z = 1 \text{ rad/s}\). A pair of attitude thrusters mounted normal to the spin axis produces a constant torque at each one-hundredth second firing. Simulate the bang-bang response to two thruster firings spaced half a precession period apart, in order to achieve the maximum spin-axis deflection.

We begin by using the approach of simulating the response to initial conditions by writing a program called spacesymnthrust.m, which is tabulated in Table 13.2. This program calculates the necessary impulse magnitudes for achieving the maximum spin-axis deviation possible with a synchronization of the inertial spin with precession, such that \(\psi = -\phi = \pi\) at the end of the second impulse. The resulting impulse magnitudes are translated into the initial conditions for the angular velocity and nutation angle, and response to the initial conditions following the impulses is simulated by solving the torque-free equations—encoded as spacesymm.m (Table 13.3)—by the MATLAB Runge–Kutta solver, \textit{ode45.m}. The MATLAB statement for invoking the program, and its effects, is given as follows:
which implies $\frac{\theta_d}{t} = 1.0472$ rad (60°) and $t_{1/2} = 4.7124$ s. The impulsive thruster torque required for this maneuver is calculated as follows:

$$M_y = \frac{J_{zz} \tan \theta_d}{\Delta t} = \frac{(500)(1) \tan 60^\circ}{0.01} = 86,602.54 \text{ s},$$

which is a rather large magnitude, considering the size of the spacecraft (e.g., a pair of thrusters symmetrically placed 2 m away from oy must produce a thrust of 21,650.64 N for 0.01 s). The resulting plots of $\omega_x(t), \omega_y(t), \omega_{xy}(t)$ and $\psi(t), \phi(t)$ are shown in Figs. 13.6 and 13.7, respectively. It is clear from these plots that the inertial spin and precession are synchronous, with both $\psi$ and $\phi$ reaching 180° simultaneously at the end of the second applied impulse ($t = t_{1/2}$). The effect of the two impulses is to instantaneously increase the angular velocity component, $\omega_y$, thereby starting and stopping precession. Since $\phi$ and $\psi$ are synchronized, the single pair of attitude thrusters firing about oy achieves the maximum possible deflection of the spin axis by 120°.

Fig. 13.6. Angular velocity response of a prolate, spin-stabilized spacecraft undergoing impulsive attitude maneuver (simulation by initial response).
Table 13.2. M-file `spacesynththrust.m` for the Simulation of Impulsive Attitude Maneuver of a Spin-stabilized Spacecraft

```matlab
%d program for rotational dynamics and Euler 3-1-3 kinematics
%d of rigid, axisymmetric, spin-stabilized spacecraft
%x(1)=omega_x, x(2)=omega_y (angular velocity in rad/s)
x(3)=psi, x(4)=phi (rad)
%u = impulsive torque about 'oy' axis (N-m)
%c 2006 Ashish Tewari
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3))
T=0.01;
n=1; %rad/s
%thd2=atan(umax*T/(n*J3))
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
```

Table 13.3. M-file `spacesymm.m` for the Torque-free Equations for a Rigid, Axisymmetric, Spinning Spacecraft

```matlab
function xdot=spacesymm(t,x)
%d program for rotational dynamics and Euler 3-1-3 kinematics
%d of rigid, axisymmetric, spin-stabilized spacecraft
% x(1)=omega_x, x(2)=omega_y (angular velocity in rad/s)
% x(3)=psi, x(4)=phi (rad)
%c 2006 Ashish Tewari
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
T=0.01; %impulse duration
thd2=acos(J3/(J1-J3));
xdot(1,1)=x(2)*n*(J1-J3)/J1;%Euler's eqn.(1)
xdot(2,1)=x(1)*n*(J3-J1)/J1;%Euler's eqn.(2)
xdot(3,1)=(sin(x(4))*x(1)+cos(x(4))*x(2))/sin(thd2); %precession rate
xdot(4,1)=n*(1-J3/J1); %inertial spin rate
```
Next we consider direct simulation using numerical integration of the spacecraft Euler’s equations and 3-2-1 Euler kinematics with bang-bang torque impulses. For this purpose, a program called \textit{spaceimpulse.m} (Table 13.4) provides the differential equations of motion to the MATLAB Runge–Kutta solver, \textit{ode45.m}. The numerical integration requires a smaller maximum time step and relative tolerance than the default values of \textit{ode45.m} because of the necessity of modeling impulsive torque. The statements for the execution of the program are given below, and the resulting plots of the state variables are shown in Figs. 13.8 and 13.9. The angular velocity response (Fig. 13.8) is identical to Fig. 13.6, whereas the 3-2-1 Euler angles produce a spin-axis deviation of $\beta = 120^\circ$, as expected at the end of the impulse sequence. It is again emphasized that this extremely large and rapid maneuver is atypical of the actual spacecraft.

\begin{verbatim}
>> options=odeset('MaxStep',0.001,'RelTol',1e-5);
>> [t,x]=ode45(@spaceimpulse,[0 6],[0 0 0 0 0]',options);
\end{verbatim}

13.6.3 Asymmetric Spacecraft Maneuvers by Attitude Thrusters

Unfortunately, the foregoing discussion of time-optimal, bang-bang control cannot be extended to a simultaneous, arbitrary rotation of an asymmetric
Fig. 13.8. Angular velocity response of a prolate, spin-stabilized spacecraft undergoing impulsive attitude maneuver (direct simulation with torque impulses).

Table 13.4. M-file `spaceimpulse.m` for State Equations of a Spin-stabilized Spacecraft with Bang-Bang Torque Impulses

```matlab
function xdot=spaceimpulse(t,x)
% Program for rotational dynamics and Euler (psi)_3 (theta)_2 (phi)_1
% Kinematics of a rigid, axisymmetric spacecraft under the
% Application of two torque impulses about 'oy' axis, spaced
% Half-precession period apart
%x(1)=omega_x, x(2)=omega_y (angular velocity in rad/s)
%x(3)=psi, x(4)=theta, x(5)=phi (rad)
%c 2006 Ashish Tewari
J1=1500; J3=500; %principal moments of inertia (kg.m^2)
thd2=acos(J3/(J1-J3));
\n% Spin rate (rad/s)
% T=0.01; % duration of impulse (s)
% umax=J3*n*tan(thd2)/T; % maximum torque of impulse (N-m)
% Ts=Ts+pi/abs(n*(1-J3/J1)); % time of application of second impulse (s)
if t>=0 && t<=T
 u=umax;
 elseif t>Ts && t<=Ts+T
 u=umax;
 else
 u=0;
 end
xdot(1,1)=x(2)*n*(J1-J3)/J1;
xdot(2,1)=x(1)*n*(J3-J1)/J1+u/J1;
xdot(3,1)=(sin(x(5))*x(2)+cos(x(5))*n)/cos(x(4));
xdot(4,1)=cos(x(5))*x(2)-sin(x(5))*n;
xdot(5,1)=x(1)+(sin(x(5))*x(2)+cos(x(5))*n)*tan(x(4));
```
spacecraft about two or three axes. This is due to the nonlinear nature of asymmetric Euler’s equations when more than one angular velocity components is nonzero, in which case the linear superposition of solutions does not hold, and the time-optimal control is not possible in a closed form. However, if the rotations are small, Euler’s equations are rendered linear by approximation, and the bang-bang approach is valid. A practical method of dealing with large, multi-axis, rest-to-rest rotations is to apply them in a sequential manner. For such an approach, attitude thrusters about any two principal axes are capable of producing an arbitrary orientation (such as the 3-1-3 Euler angle attitude representations). Of course, one may choose to fix attitude thrusters about the minor and major axes, thereby precluding the unstable intermediate axis rotation. We have already covered single-axis rotations; thus, modeling of multiple, sequential, single-axis rotations requires no further discussion.

There are advanced closed-loop control algorithms [44] for deriving thruster torques for a large and rapid maneuver of asymmetric spacecraft. Simulating the attitude response of a spacecraft to such torques with simultaneous, large, multi-axis rotations is therefore essential. Numerical integration of nonlinear, coupled Euler’s equations with applied torque and kinematic differential equations is feasible through Runge–Kutta and other iterative methods.
Let us simulate a general impulsive maneuver with the standard Runge–Kutta solver of MATLAB, \texttt{ode45.m}.

**Example 13.4.** A rigid spacecraft with principal moments of inertia 
\begin{align*}
  J_{xx} &= 400 \text{ kg.m}^2, \\
  J_{yy} &= 750 \text{ kg.m}^2, \\
  J_{zz} &= 850 \text{ kg.m}^2 \end{align*}
has three pairs of thrusters, each capable of generating a torque with adjustable magnitude and duration about a principal axis. The spacecraft is initially at rest, with initial attitude in terms of the 3-1-3 Euler angles given by \( \psi(0) = 0, \theta(0) = \frac{\pi}{2}, \phi(0) = 0 \). Simulate the attitude response of the spacecraft for 10 s to the following torque profile:

\[
M = \begin{cases}
  1000i - 1000k \text{ N.m}, & 0 \leq t \leq 1 \text{ s}, \\
  -1000i - 750j + 750k \text{ N.m}, & 5 < t \leq 5.97 \text{ s}, \\
  0, & t > 5.97 \text{ s}.
\end{cases}
\]

We begin by writing a program called \texttt{spacethruster.m} (Table 13.5) to provide the governing differential equations of motion with the specified torque to the MATLAB Runge–Kutta solver, \texttt{ode45.m}. The numerical integration is carried out with a smaller relative tolerance \((10^{-5})\) than the default value used in \texttt{ode45.m} because of the step changes in the torque. The statements for the execution of the program are given below, and the resulting plots of the state variables are shown in Figs. 13.10 and 13.11. There is a large change of attitude and angular velocity during the maneuver. At the end of the maneuver, the angular velocity becomes a near-zero constant, resulting in an almost constant attitude. It is possible to reduce the residual angular velocity to exactly zero by either using bang-bang thruster impulses as explained above, or using momentum wheels described in the next section.

\[\texttt{options=odeset('RelTol',1e-5);}\]
\[\texttt{[t,x]=ode45(@spacethruster,[0 10],[0 0 0 pi/2 0]',options);}\]

### 13.7 Spacecraft with Rotors

As the frequent use of the attitude thruster reaction control system (RCS) for stabilization and control entails a large fuel expenditure, most three-axis stabilized spacecraft additionally employ momentum exchange devices (MED), which consist of spinning rotors capable of exerting an internal torque on the spacecraft about each principal axis. As the MED are rotated by electric motors that derive their power from solar arrays of the spacecraft, they provide a fuel-free means of attitude control in the normal operation of the spacecraft. We shall consider here how a spacecraft with MED can be modeled and simulated accurately.

Consider a spacecraft with principal inertia tensor \( J \) and angular velocity resolved in the principal axes \( \omega = (\omega_x, \omega_y, \omega_z)^T \). Now consider a rotor with inertia tensor, \( J_r \), about the spacecraft’s principal axes, rotating with an angular velocity relative to the spacecraft, \( \omega_r = (\omega_{rx}, \omega_{ry}, \omega_{rz})^T \), also resolved in
Fig. 13.10. Angular velocity response of an asymmetric spacecraft to the prescribed torque profile.

Table 13.5. M-file `spacethruster.m` for State Equations of an Asymmetric Spacecraft with Specified Torque Profile

```matlab
function xdot=spacethruster(t,x)
%program for rotational dynamics and Euler 3-1-3 kinematics
%x of rigid spacecraft with arbitrary torque profile
%\dot{x}(1)=\omega_y, x(2)=\omega_z, x(3)=\omega_x (angular velocity in rad/s)
%\dot{x}(4)=\psi, x(5)=\theta, x(6)=\phi (rad)
%c 2006 Ashish Tewari
J1=400; J2=750; J3=850; %principal moments of inertia (kg.m^2)
if t>=0 && t<=1
    u=[1000;0;-1000];
elseif t>5 && t<=5.97
    u=[-1000;-750;750];
else
    u=[0;0;0];
end
xdot(1,1)=x(2)*x(3)*(J2-J3)/J1+u(1)/J1;
xdot(2,1)=x(1)*x(3)*(J3-J1)/J2+u(2)/J2;
xdot(3,1)=x(1)*x(2)*(J1-J2)/J3+u(3)/J3;
xdot(4,1)=(sin(x(6))*x(1)+cos(x(6))*x(2))/sin(x(5));
xdot(5,1)=cos(x(6))*x(1)-sin(x(6))*x(2);
xdot(6,1)=x(3)-(sin(x(6))*cos(x(5))*x(1)+cos(x(6))*cos(x(5))*x(2))/sin(x(5));
```
the spacecraft’s principal frame. The net angular momentum of the system (spacecraft and rotor) is the following:

\[ H = J \omega + J_r (\omega + \omega_r). \]  

\(^{6}\) A transformation of the inertia tensor in the rotor’s principal frame to that in the spacecraft’s principal frame can be easily performed through the parallel axes theorem. The theorem states that the inertia tensor of a mass, \(m\), about a parallelly displaced body frame, \(J\), can be derived from that in the original body frame, \(J'\), by the following expression:

\[ J = J' + m \begin{pmatrix} \Delta y^2 + \Delta z^2 & -\Delta x \Delta y & -\Delta x \Delta z \\ -\Delta x \Delta y & \Delta x^2 + \Delta z^2 & -\Delta y \Delta z \\ -\Delta x \Delta z & -\Delta y \Delta z & \Delta x^2 + \Delta y^2 \end{pmatrix}, \]

where \(\Delta x, \Delta y, \Delta z\) are the components of the parallel displacement of the body frame. After translating the principal frame of the rotor to the spacecraft’s center of mass by the parallel displacement, a rotation is performed to align the rotor’s principal axes with that of the spacecraft. If this rotation is represented by the coordinate transformation of Eq. (13.27), the inertia tensor transformed through the rotation is given by Eq. (13.31). The parallel axis theorem is also useful in deriving the inertia tensor of a complex shaped body composed of several smaller bodies with known inertia tensors.

Fig. 13.11. Attitude response of an asymmetric spacecraft to the prescribed torque profile.

the time derivative of which is zero (because no external torque acts on the system), and is written as follows:

\[
\frac{d\mathbf{H}}{dt} = (J + J_r) \frac{d\omega}{dt} + \frac{dJ}{dt} \omega + J_r \frac{d\omega_r}{dt} + \frac{dJ_r}{dt} \omega_r = 0 ,
\]

or,

\[
J \frac{d\omega}{dt} + S(\omega) J \omega = -J_r \left[ \frac{\partial (\omega + \omega_r)}{\partial t} + S(\omega) \omega_r \right] - S(\omega + \omega_r) J_r (\omega + \omega_r) ,
\]

where \( S(\omega) \) is the skew-symmetric matrix function of \( \omega \) given by Eq. (13.19).

On comparison with Euler’s equations for a rigid body [Eq. (13.32)], we see in Eq. (13.69) that the spacecraft can be treated as a rigid body, with the terms on the right-hand side treated as the torque applied by the rotor on the spacecraft. If several rotors are in the spacecraft, the right-hand side of Eq. (13.69) is replaced by a summation of the corresponding terms of all the rotors.

Equation (13.69) is a general equation for the rotation of a spacecraft with a rotor whose angular velocity can be changing in time due to a varying spin rate as well as a varying spin axis. If there is no change in the spin axis of the rotor relative to the spacecraft, the rotor’s angular momentum about a given principal axis is directly exchanged with that of the spacecraft by merely changing the rotor’s spin rate. Such a rotor with its axis fixed relative to the spacecraft is called a reaction wheel when used in a nonspin-stabilized spacecraft. When a large rotor is used to control a spin-stabilized, axisymmetric spacecraft, with its axis aligned with the spacecraft’s spin axis, the configuration is called a dual-spin spacecraft. Alternatively, if the rotor’s angular velocity relative to the spacecraft is fixed, but its axis is capable of tilting with respect to the spacecraft, thereby applying a gyroscopic torque arising out of the last term on right-hand side of Eq. (13.69) the rotor can be used to control the attitude of a nonspinning, asymmetric spacecraft. Such a rotor with a variable spin axis is called a control moment gyroscope (CMG). In some advanced spacecraft, the rotor can have a variable spin rate as well a variable axis and is called a variable-speed control moment gyroscope (VSCMG). Therefore, a VSCMG is the most general momentum exchange device, and the models for a reaction wheel and a CMG can be easily derived from it by simply neglecting some specific terms on the right-hand side of Eq. (13.69). We will briefly consider how a VSCMG and a dual-spin spacecraft can be modeled appropriately.

### 13.7.1 Variable-Speed Control Moment Gyroscope

Consider an axisymmetric rotor with a variable spin rate, mounted at a rigid spacecraft’s center of mass in such a way that its spin axis is free to rotate in all directions (Fig. 13.12). Such a rotor is termed a fully gimbaled gyroscope, and the arrangement that allows it to rotate freely about the spacecraft is
called *gimbaling*. Gimbaling can be carried out either using mechanical rotor supports hinged about the three principal axes of the spacecraft (called *gimbals*) or using a magnetic suspension. Of these, the former is more commonly employed. A motor is used to apply the necessary torque on the VSCMG rotor relative to the spacecraft about each principal axis, in order to move the rotor in a desired manner, thereby controlling the motion of the spacecraft.

Let \( M_r \) be the torque applied on the rotor. Then we can write the equations of motion of the rotor relative to the spacecraft as follows:

\[
M_r = J_r \frac{d\omega_r}{dt} + S(\omega_r)J_r \omega_r,
\]

(13.70)

where \( S(\omega_r) \) is the skew-symmetric matrix form of \( \omega_r \) given by Eq. (13.19).

The motion of the spacecraft is described by the dynamic equations, Eq. (13.69), and the kinematic equations representing the attitude. Since the instantaneous attitude of the spacecraft’s principal axes can be arbitrary, we will employ the nonsingular quaternion representation, \( q, q_4 \) (Chapter 2). The attitude kinematics of the spacecraft in terms of the quaternion are given by (Chapter 2)

\[
\frac{d}{dt}\{q, q_4\}^T = \frac{1}{2} \Omega(q(t), q_4(t))^T,
\]

(13.71)

where \( \Omega \) is the following skew-symmetric matrix of the angular velocity components:

\[
\Omega = \begin{pmatrix}
0 & \omega_z & -\omega_y \\
-\omega_z & 0 & \omega_x \\
\omega_y & -\omega_x & 0
\end{pmatrix}.
\]

(13.72)

For the general simulation of an attitude maneuver, Eqs. (13.69), (13.70), and (13.71) must be integrated in time, with given initial conditions, \( \omega(0), \omega_r(0), \) and \( q(0), q_4(0), \) and a prescribed motor torque profile, \( M(t) \). In addition, the rotor’s inertia tensor, \( J_r \), which depends on the orientation of the rotor relative to the spacecraft, must be known at the beginning of the maneuver.

**Example 13.5.** For the spacecraft with the inertia tensor and initial condition given in Example 13.2, consider the a rotor, initially at rest relative to the spacecraft, with the following inertia tensor in the spacecraft’s principal frame (not included in \( J \)):

\[
J_r = \begin{pmatrix}
50 & -10 & 0 \\
-10 & 100 & 15 \\
0 & 15 & 250
\end{pmatrix} \text{ kg.m}^2.
\]

A three-axis motion of the rotor is initiated by the application of the following motor torque profile beginning at \( t = 0 \):

\[
M_r = \begin{cases}
7i - 10j - 200k \text{ N.m}, & 0 \leq t < 5 \text{ s} , \\
-7i + 10j \text{ N.m}, & 5 < t < 10 \text{ s} , \\
0, & t \geq 10 \text{ s},
\end{cases}
\]
Simulate the response of the spacecraft for $0 \leq t \leq 40$ s.

Assuming the spacecraft and the VSCMG rotor to be rigid bodies. Neglecting friction in the rotor gimbals, we can model the system with Eqs. (13.69), (13.70), and (13.71), which are integrated in time using the Runge–Kutta algorithm of MATLAB, `ode45.m`. The time derivatives of the state variables, $\omega, \omega_r, q, \dot{q}$, are obtained from the equations of motion and are programmed in the M-file `spacevscmg.m` (Table 13.6), along with the given motor torque profile. Another program, called `skew.m` (Table 13.7), is written for evaluation the skew-symmetric form of a vector according to Eq. (13.19) within `spacevscmg.m`. The following MATLAB statements are used to specify the initial condition (through `rot313.m` and `quaternion.m` of Chapter 2) and integrate the equations of motion:

```
>> C=rot313(0.5*pi,0,0) %rotation matrix for the initial s/c attitude
C =
    1.0000       0       0
    0    0.0000       1.0000
    0   -1.0000       0.0000
>> q0=quaternion(C) %initial quaternion of s/c
q0 =
    0.707106781186547       0       0 0.707106781186547
>> [t,x]=ode45(@spacevscmg,[0 40],[0.1 -0.2 0.5 0 0 0 q0]);
```

The rotor’s relative angular velocity response and the angular velocity and attitude response of the spacecraft to the VSCMG motion are plotted in Figs. 13.13–13.15. Note that the VSCMG attains an almost constant relative speed about the principal axis $o_z$ after 5 s, with small amplitude oscillation about a mean value of $-225^\circ/s$ ($-3.93$ rad/s). The relative angular velocity components, $\omega_r\hat{x}, \omega_r\hat{y}$, however, display much larger amplitude oscillations.
Table 13.6. M-file spacevscmg.m for the Equations of Motion of a Rigid Spacecraft with a VSCMG

```matlab
function xdot=spacevscmg(t,x)
% program for torque-free rotational dynamics and quaternion kinematics
% of rigid spacecraft with a VSCMG
% x(1)=omega_rx, x(2)=omega_ry, x(3)=omega_rz (rotor relative ang. vel. (rad/s))
% x(4)=omega_x, x(5)=omega_y, x(6)=omega_z (spacecraft ang. vel. (rad/s))
% x(7)=q1, x(8)=q2, x(9)=q3, x(10)=q(4) (quaternion)
% this function needs the m-file "skew.m"
% (c) 2006 Ashish Tewari

J=diag([4000;7500;8500]); % principal inertia tensor (kg.m^2)
Jr=[50 -10 0;-10 100 15;0 15 250]; % rotor’s inertia tensor (kg.m^2)
if t>=0 && t<5
    Mr=[7;-10;-200];
elseif t>5 && t<10
    Mr=[-7;10;0];
else
    Mr=[0;0;0];
end

wr=[x(1);x(2);x(3)];
w=[x(4);x(5);x(6)];
q=[x(7);x(8);x(9);x(10)];
dwr=inv(Jr)*(Mr-skew(wr)*Jr*wr);
dw=-inv(J+Jr)*(skew(w)*J*w+Jr*(dwr+skew(w)*wr)+skew(w+wr)*Jr*(w+wr));
S=[0 w(3,1) -w(2,1) w(1,1);
   -w(3,1) 0 w(1,1) w(2,1);
   w(2,1) -w(1,1) 0 w(3,1);
   -w(1,1) -w(2,1) -w(3,1) 0];
dq=0.5*S*q;
xdot(1,1)=dwr(1,1);
xdot(2,1)=dwr(2,1);
xdot(3,1)=dwr(3,1);
xdot(4,1)=dw(1,1);
xdot(5,1)=dw(2,1);
xdot(6,1)=dw(3,1);
xdot(7,1)=dq(1,1);
xdot(8,1)=dq(2,1);
xdot(9,1)=dq(3,1);
xdot(10,1)=dq(4,1);
```

Table 13.7. M-file skew.m for the Evaluation of a Skew-symmetric Matrix

```matlab
function S=skew(v)
%S=skew-symmetric matrix, S (3x3), form of vector v (3x1)
S=[0 -v(3) v(2);v(3) 0 -v(1);-v(2) v(1) 0];
```

about mean values of 0 and $-22.5^\circ$/$s (-0.393\text{ rad/s})$, respectively. Due to this motion of the rotor, the spacecraft displays a much smoother angular velocity response in the given duration, when compared to the response of the same spacecraft to the specific initial condition without the VSCMG. This implies that a part of the spacecraft’s angular momentum is absorbed by the rotor. However, the response has by no means reached a steady state, and the angular velocity components $\omega_x$, $\omega_y$ show a divergent (unstable) behavior (Fig. 13.13). In order to study the long-term response, we plot the angular speed, $|\boldsymbol{\omega}|$, and principal angle, $\Phi$, of the spacecraft in Fig. 13.16. It is evident that the spacecraft’s rotation keeps on increasing almost steadily with
time, and the principal angle has a randomly oscillatory tendency due to the transfer of kinetic energy from the rotor. Such an unstable response is caused by the undamped motion of the VSCMG, after the motors have ceased applying a torque. In a realistic case, bearing friction would eventually bring the rotor to rest relative to the spacecraft, thereby damping spacecraft’s motion. In a practical application, the motor torque is carefully controlled in order to achieve a desired spacecraft orientation and velocity. This generally requires a feedback loop (Chapter 14) for measuring spacecraft’s attitude and angular velocity, and applying it as an input to a controller in order to generate a control torque in real time.

13.7.2 Dual-Spin Spacecraft

Often, spacecraft are required to be prolate in shape. This is because a prolate spacecraft fits neatly into the long, aerodynamically efficient payload bays of the launch vehicles. As we have seen earlier, spin about the minor axis is unstable because of internal energy dissipation. However, by using a rotor in a dual-spin configuration, the prolate spacecraft can be spin stabilized about its (minor) axis of symmetry. Such an approach is commonly employed in spin stabilizing communications satellites. Consider a prolate spacecraft with
a large rotor about its axis of symmetry, and a platform on which a communications payload is mounted (Fig. 13.17). It is required that the platform must be spinning at a very small rate (generally the rate of rotation of the planet relative to the orbit), $\omega_p$, such that the communications antennae are always pointed toward the receiving station. The net angular momentum of the dual-spin configuration in the presence of a lateral disturbance, $\omega_x, \omega_y$, is obtained from Eq. (13.67) to be

$$H = \left[J_p \omega_p + J_r (\omega_p + \omega_r)\right]k + J_{xy} (\omega_x i + \omega_y j),$$  

(13.73)

where $J_p$ is the moment of inertia of the platform about the spin axis, $J_r$ is the moment of inertia of the rotor about the spin axis, and $J_{xy}$ is the moment of inertia of the total system (platform and rotor) about the lateral (major) axis. The rotational kinetic energy of the system can be expressed as

$$T = \frac{1}{2} (J_p + J_r) \omega_p^2 + \frac{1}{2} J_r \omega_r^2 + J_r \omega_r \omega_p + \frac{1}{2} J_{xy} \omega_{xy}^2,$$  

(13.74)

where $\omega_{xy}^2 = \omega_x^2 + \omega_y^2$. Although the net angular momentum is conserved, the rotational kinetic energy is not conserved due to internal energy dissipation caused by friction between the platform and the rotor, and sloshing of the propellants in the RCS mounted on the rotor. The internal dissipation of
Fig. 13.15. Simulated attitude response of the spacecraft with a VSCMG rotor.

Fig. 13.16. Spacecraft’s angular speed and principal rotation caused by VSCMG.
kinetic energy for the platform is different from that of the rotor, and one must model each as a separate rigid body with different frictional torques. The rate of change of total rotational kinetic energy is given by

\[ \dot{T} = J_p \omega_p \dot{\omega}_p + J_r (\omega_p + \omega_r)(\dot{\omega}_p + \dot{\omega}_r) + J_{xy} \omega_{xy} \dot{\omega}_{xy}. \] (13.75)

Noting that the rate of change of angular momentum magnitude is zero, we have the following from Eq. (13.73):

\[ H \dot{H} = [J_p \omega_p + J_r (\omega_p + \omega_r)] [J_p \dot{\omega}_p + J_r (\dot{\omega}_p + \dot{\omega}_r)] + J_{xy} \omega_{xy} \dot{\omega}_{xy} = 0, \] (13.76)

from which the term pertaining to the rate of change of kinetic energy by precession can be calculated as

\[ J_{xy} \omega_{xy} \dot{\omega}_{xy} = -\frac{1}{J_{xy}} [J_p \omega_p + J_r (\omega_p + \omega_r)] [J_p \dot{\omega}_p + J_r (\dot{\omega}_p + \dot{\omega}_r)]. \] (13.77)

By substituting Eq. (13.77) into Eq. (13.75) we have

\[ \dot{T} = \dot{T}_p + \dot{T}_r, \] (13.78)

where \( \dot{T}_p \) and \( \dot{T}_r \) represent the rate of change of kinetic energy of the platform and rotor, respectively, given by

\[ \dot{T}_p = J_p [\omega_p - \frac{1}{J_{xy}} (J_p \omega_p + J_r (\omega_p + \omega_r)] \dot{\omega}_p, \] (13.79)

and

\[ \dot{T}_r = J_r [\omega_r + \omega_p - \frac{1}{J_{xy}} (J_p \omega_p + J_r (\omega_p + \omega_r)] (\dot{\omega}_p + \dot{\omega}_r). \] (13.80)
Both $\dot{T}_p$ and $\dot{T}_r$ are negative, because of internal energy dissipation due to friction and sloshing liquids. However, stability of the motion depends upon the relative magnitude of these dissipation terms, in order that the kinetic energy of precession is reduced to zero. Therefore, for stability it is crucial that the rotor provides an energy sink for the precessional motion, i.e.,

$$J_{xy} \dot{\omega}_{xy} \dot{\omega}_{xy} = (\dot{T}_p - J_p \dot{\omega}_p \dot{\omega}_p) + [\dot{T}_r - J_r (\dot{\omega}_p + \dot{\omega}_r)(\dot{\omega}_p + \dot{\omega}_r)] < 0,$$  \hspace{1cm} (13.81)

or,

$$-J_{xy} \dot{\omega}_{xy} \dot{\omega}_{xy} = [J_p \dot{\omega}_p + J_r (\dot{\omega}_p + \dot{\omega}_r)] \left[ \frac{J_p}{J_{xy}} \dot{\omega}_p + \frac{J_r}{J_{xy}} (\dot{\omega}_p + \dot{\omega}_r) \right] > 0,$$  \hspace{1cm} (13.82)

which leads to the requirement

$$J_p \dot{\omega}_p + J_r (\dot{\omega}_p + \dot{\omega}_r) > 0,$$  \hspace{1cm} (13.83)

because $\dot{\omega}_p > 0$ and $\dot{\omega}_r > 0$. Since $\dot{\omega}_p$ is small, we can neglect second-order terms involving it and its time derivative, leading to the approximations

$$\dot{T}_p \approx -\frac{J_p J_r}{J_{xy}} (\dot{\omega}_p + \dot{\omega}_r) \dot{\omega}_p,$$

$$\dot{T}_r \approx J_r \left(1 - \frac{J_r}{J_{xy}}\right) (\dot{\omega}_p + \dot{\omega}_r)(\dot{\omega}_p + \dot{\omega}_r).$$  \hspace{1cm} (13.84)

It is to be noted that both the energy dissipation terms are negative. Therefore, if the rotor is oblate ($J_{xy} < J_r$), it follows from Eq. (13.84) that $\dot{\omega}_p > 0$ and $\dot{\omega}_r > 0$. For a prolate rotor ($J_{xy} > J_r$), and $\dot{\omega}_r < 0$. Hence, the platform and an oblate rotor speed up, while a prolate rotor slows down in the presence of the lateral disturbance, $\omega_{xy}$. Thus, the stability requirement of Eq. (13.83) is unconditionally met by an oblate rotor. However, in a practical case the rotor is usually prolate, for which stability requires that

$$(J_p + J_r) \dot{\omega}_p > -J_r \dot{\omega}_r.$$  \hspace{1cm} (13.85)

In terms of the energy dissipation terms, the stability requirement for a prolate rotor is obtained by eliminating $\dot{\omega}_p$ and $\dot{\omega}_r$ from Eqs. (13.84) and (13.85), and making the assumption $\dot{\omega}_p \ll \dot{\omega}_r$:

$$-\dot{T}_p > -\dot{T}_r J_r J_{xy} - J_r.$$  \hspace{1cm} (13.86)

Hence, for a stable configuration of a prolate spacecraft with a small spin rate coupled with a prolate rotor, the platform must lose kinetic energy at a greater rate than the rotor. Due to friction between the rotor and the platform, the rotor’s spin rate decreases, and the platform speeds up, even in the absence of a lateral disturbance. If uncorrected, both rotor and platform will be eventually spinning at the same rate, which leads to an unstable configuration. In
order to prevent this, a motor is used to continually apply a small torque to the rotor bearing. Most communications satellites employ a dual-spin configuration. A recent interesting application of the dual-spin stabilization was in the *Galileo* interplanetary spacecraft of NASA. This spacecraft had an inertial (nonspinning) platform for carrying out communications with the earth during its six-year-long voyage to Jupiter, while its rotor, on which several navigational and scientific sensors were mounted, rotated at three revolutions per minute.

In summary, a prolate spacecraft is unconditionally stabilized about its minor spin axis by an oblate rotor. However, if a prolate rotor is to be used for the same purpose, the spacecraft must lose its kinetic energy at a greater rate than that of the rotor. In order to model the dynamics of a dual-spin spacecraft by differential equations, one has to apply the conservation of angular momentum [Eq. (13.68)] to the system, as well as derive Euler’s equations for the rotor alone, taking into account the internal energy dissipation by friction and sloshing.

### 13.7.3 Gravity Gradient Spacecraft

A spacecraft in a low-altitude orbit can generate an appreciable torque due to the variation of the gravity force along its dimensions, called the *gravity gradient* torque. Such a torque is considered negligible in atmospheric flight, because of the much larger aerodynamic moments. However, in space, the gravity gradient torque is large enough to exert a stabilizing (or de-stabilizing) influence over a spacecraft. The magnitude of gravity gradient can be increased by employing a long boom in the desired direction. For a large spacecraft (such as the *space station*) in low orbit, the gravity gradient torque is capable of overwhelming the attitude control system over time if not properly compensated for. This was an important reason why the *Skylab* mission came to a premature end in the 1970s. We shall model the gravity gradient dynamics and carry out a linear stability analysis for determining stable spacecraft attitudes. Consider a spacecraft in a low, circular orbit. The gravity gradient torque experienced by the craft can be written as follows:

\[
M_g = \int \rho \times g \, dm, \tag{13.87}
\]

where \( \rho \) locates an elemental mass, \( dm \), relative to the spacecraft’s center of mass (Fig. 13.18). The acceleration due to gravity, \( g \), is approximated by Newton’s law of gravitation for a spherical planet,\(^7\) and can be expanded using the binomial theorem as follows:

\(^7\)The oblateness effects have a negligible influence on the gravity gradient torque and are ignored in a linear stability analysis.
\[ g = -GM \frac{r + \rho}{|r + \rho|^3} \]
\[ = \frac{GM(r + \rho)}{r^3} \left(1 - \frac{3r \cdot \rho}{r^2} + \ldots\right), \quad (13.88) \]

where \(M\) denotes the planetary mass. Ignoring the second- and higher-order terms in Eq. (13.88), and carrying out the integral of Eq. (13.87) in terms of the body-referenced components of \(r = Xi + Yj + Zk\) and \(\rho = xi + yj + zk\) (where \(i, j, k\) are the spacecraft’s principal body axes), we have

\[ M_g = M_{gx}i + M_{gy}j + M_{gz}k, \quad (13.89) \]

where

\[ M_{gx} = \frac{3GM}{r^5}YZ(J_{zz} - J_{yy}), \]
\[ M_{gy} = \frac{3GM}{r^5}XZ(J_{xx} - J_{zz}), \quad (13.90) \]
\[ M_{gz} = \frac{3GM}{r^5}XY(J_{yy} - J_{xx}). \]

Substituting the gravity gradient torque components into Euler’s equations, Eq. (13.19), we have
\[ J_{xx} \dot{\omega}_x + \omega_y \dot{\omega}_z (J_{zz} - J_{yy}) = \frac{3GM}{r^5} YZ(J_{zz} - J_{yy}), \]
\[ J_{yy} \dot{\omega}_y + \omega_z \dot{\omega}_x (J_{xx} - J_{zz}) = \frac{3GM}{r^5} XZ(J_{xx} - J_{zz}), \]
\[ J_{zz} \dot{\omega}_z + \omega_x \dot{\omega}_y (J_{yy} - J_{xx}) = \frac{3GM}{r^5} XY(J_{yy} - J_{xx}). \] (13.91)

The equations of motion, Eq. (13.91), possess three distinct equilibrium attitudes (and their mirror images) for which any two of the angular velocity components vanish, and the third equals the orbital frequency, \( n \). Hence, one of the principal axes of the spacecraft must be normal to the orbital plane in the equilibrium attitude. Let the principal axis normal to the orbit plane be \( \mathbf{j} \).

In order to investigate the stability of the equilibrium points, we consider the general equilibrium attitude where the remaining two principal axes are along the velocity direction (\( \mathbf{i} \)) and toward the planet’s ceter (\( \mathbf{k} \)), respectively. The relative magnitudes of the principal moments of inertia, \( J_{xx}, J_{yy}, J_{zz} \), would determine the stability of the equilibrium points. We shall consider small perturbations from the general equilibrium attitude, represented by the 3-2-1 Euler angles \( \psi \) (yaw), \( \theta \) (pitch), and \( \phi \) (roll), respectively. Such an attitude representation is common in aircraft applications.

Let the equilibrium attitude of the spacecraft be given by the undisturbed body axes, \( \mathbf{i}^e, \mathbf{j}^e, \mathbf{k}^e \). The inertial angular velocity of the undisturbed triad, \( \mathbf{i}^e, \mathbf{j}^e, \mathbf{k}^e \), resolved in the instantaneous body axes, \( \mathbf{i}, \mathbf{j}, \mathbf{k} \), after a small attitude perturbation, \( \phi, \theta, \psi \), is \( n\mathbf{j}^e = n\mathbf{i} + n\mathbf{j} - n\mathbf{k} \), while the angular velocity disturbance from the equilibrium attitude is given by \( \dot{\phi} \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\psi} \mathbf{k} \). Therefore, the inertial angular velocity of the spacecraft becomes

\[ \omega = (\dot{\phi} + n\psi)\mathbf{i} + (n + \dot{\theta})\mathbf{j} + (\dot{\psi} - n\phi)\mathbf{k}. \] (13.92)

The position vector resolved in the body axes is

\[ r = r(-\sin \theta \mathbf{i} + \sin \phi \cos \theta \mathbf{j} + \cos \phi \cos \theta \mathbf{k}), \] (13.93)

which leads to \( X \approx -r\theta \), \( Y \approx r\phi \), and \( Z \approx r \) for the small perturbation, which, substituted into the Euler’s equations, Eq. (13.91), along with the angular velocity, Eq. (13.92), yield the following linearized equations of rotational motion:

\[ \ddot{\phi} = \frac{(J_{xx} - J_{yy} + J_{zz})n}{J_{xx}} \dot{\phi} - \frac{4n^2(J_{yy} - J_{zz})}{J_{xx}} \dot{\phi} \] (13.94)

\[ \ddot{\theta} = -\frac{3n^2(J_{xx} - J_{zz})}{J_{yy}} \dot{\theta}, \] (13.95)

\[ \ddot{\psi} = -\frac{(J_{xx} - J_{yy} + J_{zz})n}{J_{zz}} \dot{\psi} - \frac{n^2(J_{yy} - J_{xx})}{J_{zz}} \dot{\psi}. \] (13.96)

Clearly, the small-disturbance, linear pitching motion is decoupled from the roll-yaw dynamics and can be solved in a closed form. If \( J_{xx} > J_{zz} \), the pitching motion is a stable oscillation of constant amplitude given by
\[
\theta(t) = \theta(0) \cos n \sqrt{\frac{3(J_{xx} - J_{zz})}{J_{yy}}} t.
\]

This undamped pitching oscillation is called *libration* and requires an active damping mechanism, such as through a reaction wheel (Chapter 14). The coupled roll-yaw dynamics, Eqs. (13.94) and (13.96)—also called *nutation*—is seen to have the following characteristic equation:

\[
s^4 + n^2(1 + 3j_x + j_x j_z)s^2 + 4n^4 j_x j_z = 0,
\]

where

\[
\begin{align*}
    j_x &= \frac{J_{yy} - J_{zz}}{J_{xx}}, \\
    j_z &= \frac{J_{yy} - J_{xx}}{J_{zz}}.
\end{align*}
\]

For stability, all roots, \( s \), of the characteristic equation should have non-positive real parts (Chapter 14), which implies real and negative values of both the quadratic solutions, \( s^2 \), and leads to the following necessary and sufficient stability conditions:

\[
1 + 3j_x + j_x j_z \geq 4\sqrt{j_x j_z}, \quad j_x j_z > 0.
\]

It can be shown [2] that for a spacecraft with internal energy dissipation, the only stable gravity gradient attitude is the one with \( J_{yy} > J_{xx} > J_{zz} \), since it results in the lowest kinetic energy, apart from satisfying the stability criteria, Eq. (13.100). Thus, the minor axis should point toward (or away from) the planet’s center, while the major axis should lie along the orbit normal. Such an attitude is adopted for most asymmetric spacecraft in low orbits and is also the common attitude of the moons in our solar system. For small—or nearly axisymmetric—satellites, a long boom with an end mass can provide an effective gravity gradient stabilization.

**Example 13.6.** Consider the International Space Station (ISS) with the following inertia tensor [47]:

\[
J = \begin{pmatrix}
127908568 & 3141229 & 7709108 \\
3141229 & 107362480 & 1345279 \\
7709108 & 1345279 & 200432320
\end{pmatrix} \text{kg.m}^2.
\]

Simulate the gravity gradient motion of the ISS in a stable attitude at 93-min circular earth orbit, in response to an initial yaw-rate disturbance of \( 10^{-5} \) rad.
We begin by computing the principal inertia tensor as follows:

\[
\begin{bmatrix}
127908568 & 3141229 & 7709108 \\
3141229 & 107362480 & 1345279 \\
7709108 & 1345279 & 200432320
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.1471 & 0.9835 & 0.1052 \\
-0.9891 & 0.1460 & 0.0178 \\
0.0021 & -0.1067 & 0.9943
\end{bmatrix}
\]

\[
\begin{bmatrix}
106892554.88 & 0 & 0 \\
0 & 127538483.85 & 0 \\
0 & 0 & 201272329.17
\end{bmatrix}
\]

For a stable gravity gradient attitude, we require \( J_{xx} = 127,538,483.85 \) kg m\(^2\), \( J_{yy} = 201,272,329.17 \) kg m\(^2\), and \( J_{zz} = 106,892,554.98 \) kg m\(^2\). The coordinate transformation matrix to the principal body axes is given by \( V \) computed above. We choose to employ the complete set of nonlinear Euler equations, Eq. (13.91), along with the 3-2-1 Euler kinematics (Chapter 2) for a faithful simulation of the coupled motion. The simulation is carried out for two complete orbits using the stiff Runge–Kutta solver of MATLAB, \textit{ode23s}. The equations of motion are encoded in the M-file \textit{gravitygrad.m}, which is tabulated in Table 13.8. The response of the spacecraft is plotted in Figs. 13.19 and 13.20. The stability of the equilibrium attitude is evident, with the yaw response being of the largest angle, while roll response has the highest rate. The weak coupling between roll-yaw (nutation) and pitch (libration) motions is clear in this example. The frequency of roll oscillation is observed to be approximately 0.0016755 rad/s, which falls between the linear roll-yaw frequencies, 0.009919 rad/s and 0.00197 rad/s. The pitch and yaw
oscillations are nonharmonic due to the nonlinear coupling effects, which are significant even for the small yaw disturbance considered here.

Fig. 13.19. Angular rate response of the gravity gradient ISS to an initial yaw-rate disturbance.

13.8 Attitude Motion in Atmospheric Flight

The trajectory of an atmospheric flight vehicle is very sensitive to aerodynamic force, which are strong functions of the vehicle’s attitude relative to the flight path. Thus, rotational motion about the center of mass is crucial for atmospheric flight stability and control. When considering the rotational dynamics of aerospace vehicles within the atmosphere, one can still employ Euler’s equations, Eq. (13.18), with the assumption of a rigid vehicle, and taking into account the aerodynamic torque generated by the rotation of the vehicle, as well as a control torque applied either by the pilot, or by an automatic control system. Since the torque generated by gravity is always negligible in comparison with the aerodynamic torque, the vector $\mathbf{M}$ in Eq. (13.18) is almost entirely a sum of the aerodynamic torque and the control torque. The aerodynamic torque can be a nonlinear function of the vehicle’s attitude and angular velocity relative to the atmosphere, and can be obtained through experimental, semi-empirical, or computational fluid dynamics data. The control
torque can be generated either by aerodynamic means through the deflection of control surfaces, or by propulsive means through thrust deflection. Most atmospheric flight vehicles employ aerodynamic control torques of one kind or another, due to the ease by which such torques can be created. However, there are certain flight situations where an aerodynamic control torque is infeasible, such as the vertical take-off of airplanes and launch vehicles, and the initial phase of atmospheric entry, wherein the dynamic pressure is not large enough to create a sufficient control torque. Moreover, in certain highly agile missiles and fighter airplanes, the vehicle’s design precludes the generation of required control torque purely by aerodynamic means. In all such cases, thrust vectoring is employed by rotating the thrust vector relative to the body axes, in order to create the required control torque.

Since most atmospheric flight vehicles are designed to operate efficiently with a low drag, their attitude maneuvers do not create large flow disturbances in normal operation. Therefore, the assumptions of small-disturbance aerodynamics (Chapter 10) remain valid during a general attitude maneuver within the atmosphere. However, there are special circumstances where the small-disturbance approximation is invalid, namely the separated flowfield of a stalled flight, strong normal shock waves during transonic flight, and strong viscous interactions and entropy gradients in hypersonic flight. In such cases, the aerodynamic forces and moments must be derived through
wind-tunnel tests, flight tests, or by advanced computational fluid dynamic models of the nonlinear, turbulent flow. It is beyond the scope of this book to discuss modeling of nonlinear aerodynamic phenomena. We shall generally follow the common practice of employing linearized aerodynamics that results from the assumption of small disturbances in the flow field. Wherever such an approximation cannot be applied (such as post-stall maneuvers of fighter aircraft, rolling missiles, and atmospheric entry vehicles), we shall either employ simple empirical methods, or experimental aerodynamic data.

13.8.1 Equations of Motion with Small Disturbance

The governing equations of rotational motion of a rigid vehicle during atmospheric flight consist of Euler’s equations with aerodynamic and propulsive moments, kinematic equations of rotational motion, as well as the dynamic and kinematic equations of translation. The latter are necessary because the aerodynamic moments depend upon the relative velocity through the atmosphere, as well as the position (altitude) within the atmosphere. Therefore, it would appear that a six-degree-of-freedom simulation is indispensible for a flight vehicle. However, when employing the small-disturbance theory, a simplification of equations of motion results, enabling the de-coupling of the degrees of freedom, as seen below.

Let us begin with the vehicle initially in a steady, flight dynamic equilibrium, with planet-centered position, \( r^e, \delta^e, \omega^e \), and relative velocity in the local horizon frame, \( v^e, \phi^e, \lambda^e \). This equilibrium condition is chosen such that the velocity of the center of mass relative to the atmosphere is a constant, and the angular velocity components of the vehicle about the center of mass, referred to a body-fixed frame, are time-invariant. Such an equilibrium condition could be an unaccelerated, rectilinear flight, or a steady, curved flight (steady coordinated turn, steady roll, entry trajectory, etc.). In this regard, our treatment of small-disturbance rotational motion is more general than the rectilinear flight equilibrium commonly found in textbooks on flight stability and control [45], [46]. The equilibrium condition generates a reference trajectory about which the vehicle’s rotation is to be studied, after a small flow disturbance is applied to the vehicle at some time, taken to be \( t = 0 \). The equilibrium prevailing immediately before the disturbance is called the equilibrium point. The aerodynamic force and moment vectors (and their components), as well as the state variables, at the equilibrium point are denoted by the superscript \( e \), whereas the quantities immediately following the application of the disturbance, are denoted by prime. The disturbances themselves are indicated by normal symbols. A disturbed quantity, such as the relative velocity, \( v' \), is thus written as

\[
v' = v^e + \Delta v .
\]  \hspace{1cm} (13.101)

The flow-field disturbance applied at \( t = 0 \) causes an instantaneous deflection of the relative velocity vector and serves as the initial condition for the
vehicle’s motion. In order to study the stability of the equilibrium point, it is sufficient to study the vehicle’s response to a small disturbance, which, as pointed out above, is easier to model than that of a large flow disturbance. The primary objective of the rotational stability analysis is, thus, to model the small-disturbance attitude motion caused by an instantaneous change in the relative velocity. The attitude motion, in turn, causes a change in the external force and moment. It must be clear that instead of considering the response of the flight vehicle to the application of an external force and moment, we are interested in the changes in the external force and moment caused by a small disturbance in the vehicle’s velocity, which results in a rotational motion of the vehicle. If the ensuing motion beginning from a given equilibrium point is such that the flow-disturbance increases with time, we have an unstable equilibrium point. On the other hand, if the changes in the external force and moment caused by the rotational motion tend to alleviate the disturbance, the equilibrium point is said to be stable. Consider a body-fixed frame \((oxyz)\) with origin at the vehicle’s center of mass such that the axis \(ox\) along the instantaneous relative velocity vector at the equilibrium point. The axes of \(oxyz\) can thus be chosen to be parallel to the wind axes, \((Sx_s, y_s, z_s)\) (Chapter 12), at the equilibrium point, \(t = 0\). Such a coordinate system, depicted in Fig. 13.21, is referred to as the stability axes, and is quite useful in representing aerodynamic force and moment, as well as in analyzing the stability of the rotational motion. The instantaneous rotation of the velocity vector caused by the applied flow disturbance leads to the displaced wind axes, whose orientation can be described relative to the stability axes using the 3-2-1 Euler
angles, $C = C_1(\sigma)C_2(\alpha)C_3(\beta)$, as shown in Fig. 13.21, where $\sigma, \alpha, \beta$ denote the changes in the aerodynamic bank angle, the angle of attack, and the sideslip angle, respectively. Therefore, the instantaneous changes in the flight-path angle and the velocity azimuth are $\phi = \alpha$ and $A = \beta$, and the velocity vector, immediately after the flow disturbance at $t = 0$, is given by

$$v' = v^e + v,$$
$$\phi' = \phi^e + \alpha,$$  \hspace{1cm} (13.102)
$$A' = A^e + \beta.$$  

The quantities $v, \alpha, \beta$ are to be regarded as the instantaneous flow disturbance, to which the rotational response is desired. The instantaneously displaced wind axes brought to the center of mass, $ox'y'z'$, are depicted in Fig. 13.21. The coordinate transformation between the stability and wind axes is given by

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = C \begin{bmatrix} i \\ j \\ k \end{bmatrix},$$  \hspace{1cm} (13.103)

where

$$C = \begin{pmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ (\sin \sigma \sin \alpha \cos \beta - \cos \sigma \sin \beta) & (\sin \sigma \sin \alpha \sin \beta + \cos \sigma \cos \beta) & \sin \sigma \cos \alpha \\ (\cos \sigma \sin \alpha \cos \beta + \sin \sigma \sin \beta) & (\cos \sigma \sin \alpha \sin \beta - \sin \sigma \cos \beta) & \cos \sigma \cos \alpha \end{pmatrix}.$$  \hspace{1cm} (13.104)

Since the flow disturbance is small, we may assume the angles $\sigma, \alpha, \beta$ to be small, such that $\sin \alpha \approx \alpha$, $\cos \alpha \approx 1$, etc., and ignore products of angles. This leads to the following skew-symmetric approximation of the rotation matrix:

$$C \approx \begin{pmatrix} 1 & \beta & -\alpha \\ -\beta & 1 & \sigma \\ \alpha & -\sigma & 1 \end{pmatrix}.$$  \hspace{1cm} (13.105)

The kinematic relationship between the disturbance caused in the direction of the velocity vector (given by $\phi, A$), and the flow-disturbance angles, $\alpha, \beta$, is then derived as follows:

$$v' \approx v^e i + vi + v^e A j - v^e \phi k$$
$$= (v^e + v)i' = (v^e + v)(i + \beta j - \alpha k)$$
$$\approx (v^e + v)i + v^e \beta j - v^e \alpha k,$$  \hspace{1cm} (13.106)

from which it follows that $\phi \approx \alpha$ and $A \approx \beta$.

The net translational acceleration of the center of mass relative to the wind axes was derived in Chapter 12, whose equilibrium and disturbed values are denoted here by $a_{v^e}$ and $a_{v'}$, respectively, such that
\[ a_v' = a_v^e + a_v. \] (13.107)

However, it is desired to express all motion variables in the stability axes, which is a body-fixed frame. Let \( \omega^e \) be the angular velocity of the stability axes relative to the wind axes at the equilibrium point. Following the usual aeronautical nomenclature of roll rate, \( P^e = \omega^e_x \), pitch rate, \( Q^e = \omega^e_y \), and yaw rate, \( R^e = \omega^e_z \), we can express the disturbed angular velocity of the vehicle about its center of mass referred to the stability axes as

\[ \omega' = P'i + Q'j + R'k \]
\[ = (P^e + P)i + (Q^e + Q)j + (R^e + R)k \]
\[ = P'^e i + Q'^e j + R'^e k + P i + Q j + R k \] (13.108)

where \( P, Q, R \) are the angular rate disturbances. The translational acceleration at equilibrium point, referred to the stability axes is then obtained as follows:

\[ a^e = a_v^e - \omega^e \times v^e \]
\[ = a_v^e - (P^e i + Q^e j + R^e k) \times (v^e i) \] (13.109)
\[ = a_v^e - v^e(R^e j - Q^e k). \]

Similarly, the disturbed translational acceleration referred to the stability axes is given by

\[ a' = a_v' - \omega' \times v' \]
\[ = a_v' - (P'i + Q'j + R'k) \times (v'i_v') \]
\[ = a_v' - v'^e[-(\alpha Q^e + \beta R^e)i + (\alpha P^e + R^e + R)j] \]
\[ + (\beta P^e - Q^e - Q)k + v(Q^e k - R^e j) \] (13.110)

where the small-disturbance assumption has been made. Finally, the disturbance translational acceleration is obtained by subtracting Eq. (13.109) from Eq. (13.110) as

\[ a = a' - a^e \]
\[ = a_v - v'^e[-(\alpha Q^e + \beta R^e)i + (\alpha P^e + R)j + (\beta P^e - Q)k] \]
\[ + v(Q^e k - R^e j). \] (13.111)

Another kinematic relationship is possible by considering the angular velocity of the stability axes relative to the instantaneous wind axes. This difference in the angular velocities of the two frames can be written as \( \omega = Pi + Qj + Rk \), where \( Q, R \) are the differential pitch and yaw rates due to the relative rotation. We can differentiate Eq. (13.103) to obtain (Chapter 2)

\[ \frac{dC}{dt} = -CS(\omega), \] (13.112)
where
\[
S(\omega) = \begin{pmatrix} 0 & -\bar{R} & \bar{Q} \\ \bar{R} & 0 & -\bar{P} \\ -\bar{Q} & \bar{P} & 0 \end{pmatrix},
\]
resulting in
\[
\dot{\sigma} \approx P. \tag{13.114}
\]

Here, we have chosen not to express the time derivatives of \(\alpha, \beta\) in terms of the unknown variables \(\bar{Q}, \bar{R}\), which have to be obtained from the solution of the combined translation and rotational equations of motion. Instead, we can derive these derivatives in the following manner.

It is our objective to derive the time derivatives of the velocity components from the disturbance translational dynamic equation of motion, expressed in the stability axes as follows:
\[
f = ma, \tag{13.115}
\]
where \(f = f' - f^e\) is the net disturbance force resolved is the stability axes. From Chapter 12, it is clear that the net external force is a vector sum of the gravity, aerodynamic, and thrust forces. It can be generally assumed that the changes in the position, \(r, \delta, l\), are negligible during the small-disturbance motion. Thus, we have
\[
r' \approx r^e, \quad \delta' \approx \delta^e, \quad l' \approx l^e,
\]
\[
g^e_c \approx g^e_c, \quad g^k \approx g^k_0. \tag{13.116}
\]

These assumptions make the magnitude of the gravity force essentially unchanged by the small disturbance. However, its components resolved in the stability axes are functions of the disturbances. It is also to be noted that the gravitational components depend upon the instantaneous vehicle attitude relative to the local horizon, and are independent of the translatory motion represented by \(\alpha, \beta\). Therefore, it is necessary to model the gravity disturbance in terms of the stability axes rotation, such as through the 3-2-1 Euler angles, \(\Psi\) (yaw angle), \(\Theta\) (pitch angle), and \(\Phi\) (roll angle), representing the change in the body attitude relative to a north, east, down (NED) triad, \(\mathbf{I}, \mathbf{J}, \mathbf{K}\). Since \(\Phi, \Theta, \Psi\) are small, the Euler angle singularity (Chapter 2) is avoided. The coordinate transformation between the stability axes and the NED local horizon frame is given by
\[
\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = C_1(\Phi)C_2(\Theta)C_3(\Psi) \begin{pmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{pmatrix}. \tag{13.117}
\]

The acceleration due to gravity in the NED frame is the following:
\[
\frac{f^e_c}{m} = g^e = g^e_\mathbf{K} + g^e_\mathbf{L}. \tag{13.118}
\]
Using the small-disturbance approximation, the gravity disturbance can be resolved in the stability axes as follows:

\[
g = g' - g^e = g^e \left[ -\Theta \cos \Theta^e i + (\Phi \cos \Phi^e \cos \Theta^e - \Theta \sin \Phi^e \sin \Theta^e) j \\
- (\Theta \cos \Phi^e \sin \Theta^e + \Phi \sin \Phi^e \cos \Theta^e) k \right] + g^e \left[ -(\Theta \sin \Theta^e \cos \Psi^e + \Psi \cos \Theta^e \sin \Psi^e) i \right.
+ (\Phi \cos \Phi^e \sin \Theta^e \cos \Psi^e + \Theta \cos \Theta^e \sin \Phi^e \cos \Psi^e - \Psi \sin \Phi^e \sin \Theta^e \sin \Psi^e + \Phi \sin \Phi^e \sin \Psi^e - \Psi \cos \Phi^e \cos \Psi^e) j
+ (\Phi \sin \Phi^e \sin \Psi^e + \Psi \sin \Phi^e \cos \Psi^e) k \right] .
\]

(13.119)

In the derivation of Eq. (13.119)—and in other following derivations—we have used the approximations in the trigonometric terms involving small-disturbance angles, such as

\[
\sin(\Phi^e + \Phi) = \sin \Phi^e \cos \Phi + \cos \Phi^e \sin \Phi \approx \sin \Phi^e + \Phi \cos \Phi^e ,
\cos(\Psi^e + \Phi) = \cos \Phi^e \cos \Phi - \sin \Phi^e \sin \Phi \approx \cos \Phi^e - \Phi \sin \Phi^e .
\]

(13.120)

It is to be noted that the equilibrium attitude of the vehicle is that of the undisturbed stability axes and is given by \( \Theta^e = \phi^e, \Psi^e = \lambda^e \).

The sum of disturbed aerodynamic and propulsive force vectors, resolved in the displaced wind axes, is the following:

\[
f_a' + f_T' = (f_T' \cos \epsilon^e \cos \mu^e - D') i'
+(f_Y' + f_T' \sin \mu^e) j'
-(f_T' \sin \epsilon^e \cos \mu^e + L') k',
\]

(13.121)

where we have assumed that the equilibrium thrust angles, \( \epsilon^e, \mu^e \), are unchanged by the flow disturbance. This is true for most well-designed vehicles, where the thrust is generated either by aerodynamic means or by a rocket engine having a freely swiveling nozzle that always maintains a fixed orientation relative to the wind axes. The equilibrium sum of the aerodynamic and propulsive forces is

\[
f_a^e + f_T^e = (f_T^e \cos \epsilon^e \cos \mu^e - D^e) i
+(f_Y^e + f_T^e \sin \mu^e) j
-(f_T^e \sin \epsilon^e \cos \mu^e + L^e) k.
\]

(13.122)

Employing the small-disturbance approximation, the disturbance force arising out of aerodynamics and propulsion is written as:
We remind ourselves that the aerodynamic and propulsive disturbance terms, \(L, D, f_Y, f_T\), depend upon the flow disturbances, \(\alpha, \beta, \sigma\). We shall express the linearized relationships of these disturbance terms a little later.

It now remains to obtain an expression for the disturbance translational acceleration of the center of mass, \(a_v\), resolved in the stability axes. In order to do so, we shall first write the translational acceleration at equilibrium by substituting \(\dot{v} = \dot{A} = 0\) into the acceleration derived in Chapter 12, leading to

\[
\begin{align*}
a_v^e &= a_x^e i + a_y^e j + a_z^e k \\
&= v^e \left[ -\frac{v^e}{r^e} \cos^2 \phi^e \sin A^e \tan \delta^e \\
&\quad + 2 \Omega^e (\sin \phi^e \cos A^e \cos \delta^e - \cos \phi^e \sin \delta^e) \right] j \\
&\quad + \left( \frac{v^e}{r^e} \cos \phi^e + 2 \Omega^e \sin A^e \cos \delta^e \right) k,
\end{align*}
\]

where the centripetal acceleration terms due to planetary rotational velocity, \(\Omega\), are neglected, as they are several orders of magnitude smaller than the other terms. However, we shall (for the time being) retain the Coriolis acceleration terms due to planetary rotation. These are generally negligible for most atmospheric vehicles, except an atmospheric entry vehicle.\(^8\) The disturbance acceleration, \(a_v\), is obtained as follows by subtracting Eq. (13.65) from the disturbed acceleration, \(a_v'\), resolved in stability axes, and applying the small-disturbance approximation:

\[
\begin{align*}
a_v &= (\dot{v} - \beta a_y^e + \alpha a_z^e) i \\
&\quad + \left[ v^e \beta + \frac{v^e}{r^e} \cos \phi^e \tan \delta^e (v^e \alpha \sin \phi^e \sin A^e \\
&\quad - v^e \beta \cos \phi^e \cos A^e - 2 v^e \cos \phi^e \sin A^e) \right] j
\end{align*}
\]

\(^8\) For a typical entry from a low earth orbit, \(v^e = 8 \text{ km/s}\) and \(r^e = 6500 \text{ km}\). This yields the maximum centripetal acceleration, \(\Omega^2 r^e \approx 0.03 \text{ m/s}^2\), maximum Coriolis acceleration, \(2 \Omega v^e \approx 1 \text{ m/s}^2\), and maximum acceleration due to planetary curvature, \(\frac{v^e}{r^e} \approx 10 \text{ m/s}^2\). Thus, curvature and Coriolis acceleration cannot be ignored, as they are of the same order, and one tenth, respectively, of the magnitude of acceleration due to gravity. The centripetal acceleration terms vanish below first order, when multiplied with a small disturbance, and are thus neglected in a stability analysis.


\[ + 2\Omega \{ v \cos \delta^c \sin \phi^c \cos A^e - \sin \delta^c \cos \phi^c \} \]
\[ + v^c \alpha (\cos \delta^c \cos \phi^c \cos A^e + \sin \delta^c \sin \phi^c) \]
\[ - v^c \beta \cos \delta^c \sin \phi^c \sin A^e \} + \beta a_{xy}', - \sigma a_{xy} \} \]
\[ + \left[ v^c \dot{\alpha} + \frac{v^c}{\rho^e} (2v \cos \phi^e - v^c \alpha \sin \phi^e) \right] - \alpha a_{xy}' + \sigma a_{xy} \left[ k \right]. \]

Collecting all the terms from Eqs. (13.125), (13.119), (13.123), and (13.111), and substituting them into Eq. (13.115), we have the disturbance force equations:

\[ f_T \cos \epsilon^c \cos \mu^e - D - \beta (f_T' + f_T'' \sin \mu^e) - \alpha (f_T' \sin \epsilon^c \cos \mu^e + L^e) \]
\[ - m g^e \cos \theta^e \cos \epsilon^c \cos \psi^e + \Psi \cos \theta^e \sin \psi^e \}
\[ = m [v^c \dot{\beta} + \frac{v^c}{\rho^e} \cos \phi^e \tan \delta^c (v^c \alpha \sin \phi^e \sin A^e) \]
\[ - v^c \beta \cos \phi^e \cos A^e - 2v \cos \phi^e \sin A^e \}
\[ + 2\Omega \{ v \cos \delta^c \sin \phi^e \cos A^e - \sin \delta^c \cos \phi^e \} \]
\[ + v^c \alpha (\cos \delta^c \cos \phi^e \cos A^e + \sin \delta^c \sin \phi^e) \]
\[ - v^c \beta \cos \delta^c \sin \phi^e \sin A^e \} + \beta a_{xy}' \]
\[ - \sigma a_{xy}' - v^c (\alpha P^e + R) - v R^e \right] \].
\[ - f_T' \sin \epsilon^c \cos \mu^e - L + \sigma (f_T' + f_T'' \sin \mu^e) \]
\[ - m g^e \Theta \cos \phi^e \sin \theta^e + \Phi \sin \phi^e \cos \theta^e \]
\[ + m g^e \Theta \cos \phi^e \sin \theta^e \sin \psi^e \]
\[ + \Theta \cos \phi^e \sin \theta^e \sin \psi^e + \sin \phi^e \sin \theta^e \cos \theta^e \]
\[ + \Phi \cos \phi^e \sin \psi^e + \Psi \sin \phi^e \cos \psi^e \}
\[ = m \left[ - v^c \dot{\alpha} + \frac{v^c}{\rho^e} (2v \cos \phi^e - v^c \alpha \sin \phi^e) \right] - \alpha a_{xy}' + \sigma a_{xy}' - v^c (\beta P^e - Q) + v Q^e \right] \].
These equations will be further expanded when we take into account the linear variation of the aerodynamic and thrust forces with the flow disturbances.

At equilibrium, the vehicle is rotating with a constant, body-referenced angular velocity, \( \omega^e = P^e \hat{i} + Q^e \hat{j} + R^e \hat{k} \), and equilibrium torque, \( \mathbf{M}^e = \mathbf{L}^e \hat{i} + \mathbf{M}^e \hat{j} + \mathbf{N}^e \hat{k} \), where \( \mathbf{L}^e \) is called the rolling moment, \( \mathbf{M}^e \) the pitching moment, and \( \mathbf{N}^e \) the yawing moment—in standard aeronautical nomenclature—at the equilibrium point. Therefore, Euler’s equations of rotational motion Eq. (13.18), expressed in the stability axes at equilibrium, yield the following equations for the torque components at equilibrium:

\[
\begin{align*}
\mathbf{L}^e &= Q^e [R^e (J_{zz} - J_{yy}) - P^e J_{xz} - Q^e J_{yz}] + R^e (P^e J_{xy} + R^e J_{yz}), \\
\mathbf{M}^e &= P^e [R^e (J_{xx} - J_{zz}) + Q^e J_{yz} + P^e J_{zz}] - R^e (Q^e J_{xy} + R^e J_{zz}), \\
\mathbf{N}^e &= Q^e [P^e (J_{yy} - J_{xx}) + Q^e J_{xy} + R^e J_{zz}] - P^e (R^e J_{xy} + P^e J_{yy})
\end{align*}
\]

(13.129)

where we note the presence of products of inertia, which are nonzero because \((oxyz)\) is not the principal frame. Most atmospheric flight vehicles possess a plane of symmetry, and often the equilibrium flight condition is such that the velocity vector lies in the plane of symmetry. Such an assumption would greatly simplify Eq. (13.129). However, we shall reserve this assumption for later, because a general flight path may not obey this restriction (e.g., the steady sideslip maneuver of aircraft). In a manner similar to the disturbance force, we can derive the disturbance torque components by subtracting the equilibrium torque from the disturbed torque, resulting in

\[
\begin{align*}
\mathbf{L} &= J_{xz} \dot{P} - J_{xy} \dot{Q} - J_{zz} \dot{R} - J_{xz} (QP^e + PQ^e) + 2J_{yz}(RR^e - QQ^e) \\
&+ J_{xy} (RP^e + PR^e) + (J_{zz} - J_{yy})(QR^e + RQ^e). \\
\mathbf{M} &= J_{yy} \dot{Q} - J_{xy} \dot{P} - J_{yz} \dot{R} - J_{xy} (QP^e + PQ^e) + 2J_{yz}(PP^e - RR^e) \\
&+ J_{yz} (QP^e + PQ^e) + (J_{xx} - J_{zz})(RP^e + PR^e). \\
\mathbf{N} &= J_{zz} \dot{R} - J_{xz} \dot{P} - J_{yz} \dot{Q} - J_{zz} (RP^e + PR^e) + 2J_{xy}(QQ^e - PP^e) \\
&+ J_{xz} (RP^e + PR^e) + (J_{yy} - J_{xx})(QP^e + QP^e).
\end{align*}
\]

(13.130)

(13.131)

(13.132)

The rate of rotation of the stability axes is affected by the disturbance torque, which in turn, is changed by the flow disturbance. However, due to the rotary inertia of the vehicle, the change in the vehicle’s attitude is not instantaneous, but occurs over a period of time. Since the attitude of the vehicle is described by the orientation of the body-fixed stability axes, an appropriate representation can be used for the instantaneous orientation of \((oxyz)\). If the vehicle is initially at rest, the 3-2-1 Euler angle representation would be non-singular during the rotational motion caused by the small disturbance. For this reason, the 3-2-1 body attitude representation is most popular in aircraft applications. In such a case, the vehicle’s attitude at \( t = 0 \) is given by the equilibrium attitude, \( \Psi^e, \Theta^e, \Phi^e \), while the perturbation from this attitude is given by the
Disturbance angles, $\Psi, \Theta, \Phi$. However, when the equilibrium state is a general rotary motion, a more appropriate attitude description is via the quaternion (Chapter 2), $q, q_4$, whose kinematic equations are written as follows:

$$\frac{d[q, q_4]^T}{dt} = \frac{1}{2}\Omega(q(t), q_4(t))^T,$$

(13.133)

where $\Omega$ is the following skew-symmetric matrix:

$$\Omega = \begin{pmatrix}
0 & (R^e + R) & -(Q^e + Q) & (P^e + P) \\
-(R^e + R) & 0 & (P^e + P) & (Q^e + Q) \\
(Q^e + Q) & -(P^e + P) & 0 & (R^e + R) \\
-(P^e + P) & -(Q^e + Q) & -(R^e + R) & 0
\end{pmatrix}.$$  

(13.134)

The kinematic equation is to be integrated with the initial condition specified by the equilibrium point attitude, $q^e, q^e_4$.

The equations of a small-disturbance, rotational motion of an atmospheric flight vehicle, therefore, consist of the coupled set of differential equations, Eqs. (13.114), (13.126)–(13.128), (13.131)–(13.134). The additional coupling terms due to aerodynamics and propulsion in these equations are derived by the linearized stability derivatives.

**13.8.2 Stability Derivatives and Decoupled Dynamics**

The stability analysis of an atmospheric vehicle requires the functional dependence of the aerodynamic and propulsive force and moment on the disturbance variables. In a general unsteady motion of a flight vehicle, such relationships are non-existent in a closed form, due to the complex effects of turbulence, compressibility, flow separation, and non-continuity. Even when simplifying assumptions are made, rarely do we have a closed-form description of the flow field (Chapter 10), and an approximate, numerical solution of partial differential equations is the norm. However, one can employ the small-disturbance approximation to render all aerodynamic relationships essentially linear, irrespective of the flow regime in which the equilibrium point is located. The hallmark of linear dependence of the aero-propulsive force and moment on the disturbance variables is a Taylor series expansion, truncated to first-order terms, such as the following expression for the disturbed pitching moment, $M'$, of an airplane:

$$M' = M^e + \frac{\partial M}{\partial v}v + \frac{\partial M}{\partial \alpha} \alpha + \frac{\partial M}{\partial \dot{\alpha}} \dot{\alpha} + \frac{\partial M}{\partial Q} Q,$$

(13.135)

where the partial derivatives are evaluated at the equilibrium point and are referred to as stability derivatives. It is useful to express the stability derivatives in a nondimensional form, which allows us to analyze the characteristics of a particular configuration without having to consider the effects of size, equilibrium speed, and altitude. This is accomplished by dividing the forces
by \( qS \), moments by \( qSl_c \), and speed by \( v^e \), where \( q \) is the dynamic pressure, \( S \) is the reference wing planform area, and \( l_c \) is a characteristic length. The angular rates are traditionally expressed in a nondimensional time, \( \hat{t} = t \frac{v^e}{l_c} \), which results in the corresponding nondimensional stability derivatives being multiplied by the factor, \( \frac{1}{l_c^2 v^e} \), in the equations of motion. For example, the nondimensionalized pitching moment disturbance can be expressed using Eq. (13.135) in the standard NACA nomenclature as follows:

\[
C_m = C_{m_u} u + C_{m_\alpha} \alpha + \frac{\bar{e}}{2v^e} C_{m_\dot{\alpha}} \dot{\alpha} + \frac{\bar{e}}{2v^e} C_{m_Q} Q,
\]

(13.136)

where

\[
C_{m_u} = \frac{\bar{v}^e}{qS\bar{c}} \frac{\partial M}{\partial v},
C_{m_\alpha} = \frac{1}{qS\bar{c}} \frac{\partial M}{\partial \alpha},
C_{m_\dot{\alpha}} = \frac{2v^e}{qS\bar{c}} \frac{\partial M}{\bar{c} \partial \dot{\alpha}},
C_{m_Q} = \frac{1}{qS\bar{c}} \frac{\partial M}{\bar{c} \partial Q},
\]

(13.137)

and \( \bar{c} \) is the wing’s mean-aerodynamic chord, representing the characteristic length. The characteristic length (and thus the nondimensional time) indicates the time scale of motion and may be different for the various stability axes. Furthermore, the force and moment relative to each stability axis may depend upon a different set of disturbance quantities. Before pursuing the concept of stability derivatives any further, it is important to define the set of motion variables particular to each stability axis.

All atmospheric flight vehicles possess some form of symmetry, which enables them to achieve a stable equilibrium in normal operation. The least symmetric atmospheric flight vehicle is a lifting configuration—such as the airplane—having only one plane of symmetry, \( oxz \), while a thrust-controlled missile is an axisymmetric vehicle with infinitely many planes of symmetry (and thus non-unique stability axes). In between these two extremes lie most launch vehicles and missiles, with a varying number of symmetry planes. A vehicle with more than one plane of symmetry enjoys inter-exchangeability of two (or more) stability axes, for which the equations of motion are identical. Therefore, the airplane is taken to be the reference vehicle for defining the dependent motion variables for each stability axis, as it results in the most general description of aerodynamic motion. Assuming \( oxz \) to be a plane of symmetry, we have \( J_{xy} = J_{yz} = 0 \). Furthermore, it follows that the motion in the plane of symmetry, called longitudinal dynamics, is fundamentally different—and separable from—that outside the plane, which we will refer to
as lateral dynamics. Hence, longitudinal and lateral dynamics should have distinct sets of motion variables. Clearly, the lateral dynamics involves changes in the “unsymmetrical” variables $\beta, \Phi, \Psi, P, R$, whereas longitudinal motion involves the remaining variables, namely $u, \alpha, \Theta, Q$. With these assumptions, we can de-couple the longitudinal and lateral dynamics and separate the stability derivatives into the two categories.

### 13.8.3 Longitudinal Dynamics

The longitudinal dynamic equations—Eqs. (13.126), (13.128), and (13.131)—involve a three-degree-of-freedom motion (translation along $o_x, o_z$, and rotation about $o_y$). For a flight in the plane of symmetry, $\beta = \Phi = \Psi = \Phi^e = P^e = R^e = 0$. Hence, the longitudinal equations of motion are written in the following de-coupled form:

$$m[v^e \dot{u} + \alpha(a^e_v + v^e Q^e)] = qS[C_{xu} u + C_{xa} \alpha - \Theta \left(\cos \Theta \frac{m g^e}{q S} + \sin \Theta \cos \psi^e \frac{m g^e}{q S}\right) + \frac{c}{2\nu^e} (C_{xa} \dot{\alpha} + C_{xa} Q)].$$  \hspace{1cm} (13.139)

$$m[v^e \ddot{\alpha} - \frac{\nu^e}{\nu^e} (2v \cos \phi^e - v^e \alpha \sin \phi^e) - 2u \nu^e \Omega \cos \delta^e \sin \alpha^e + \alpha u^e - u \nu^e Q^e - Qv^e]] = qS[C_{zu} u + C_{za} \alpha - \Theta \left(\sin \Theta \frac{m g^e}{q S} - \cos \Theta \cos \psi^e \frac{m g^e}{q S}\right) + \frac{c}{2\nu^e} (C_{za} \dot{\alpha} + C_{za} Q)].$$  \hspace{1cm} (13.140)

$$J_{yy} \dot{Q} = qS[c_{m_u} u + C_{m_\alpha} + \frac{\nu^e}{2\nu^e} (C_{m_u} \dot{\alpha} + C_{m_\alpha} Q)].$$  \hspace{1cm} (13.141)

Upon comparison with Eqs. (13.126) and (13.128), some of the longitudinal force derivatives are directly obtained to be the following:

$$C_{xu} = 2(C_T \cos \epsilon^e - C_D) - u \frac{\partial C_D}{\partial \alpha} + u \cos \epsilon^e \frac{\partial C_T}{\partial u}.$$  \hspace{1cm} (13.142)

$$C_{xa} = -C_T \sin \epsilon^e - C_L - \frac{\partial C_D}{\partial \alpha} + \cos \epsilon^e \frac{\partial C_T}{\partial \alpha},$$  \hspace{1cm} (13.143)

$$C_{zu} = -2(C_T \sin \epsilon^e + C_L) - u \frac{\partial C_L}{\partial \alpha} - u \sin \epsilon^e \frac{\partial C_T}{\partial u},$$  \hspace{1cm} (13.144)

$$C_{za} = C_T \cos \epsilon^e - C_D - \frac{\partial C_L}{\partial \alpha} - \sin \epsilon^e \frac{\partial C_T}{\partial \alpha}.$$  \hspace{1cm} (13.145)

Here, $C_L, C_D, C_T$ refer to the lift, drag, and thrust coefficients (Chapters 10 and 11). Except for propeller-engined airplanes, the variation of $C_T$ with speed and angle of attack is negligible. The variation of drag coefficient with
speed occurs due to compressibility effects and is especially important in the transonic regime (Chapter 10). Hence, we commonly calculate such derivatives using the Mach number, \( M \), as

\[ u \frac{\partial C_D}{\partial u} = M \frac{\partial C_D}{\partial M}. \]

Generally, a well-designed airplane has \( C_{m_\alpha} \approx 0 \).

\( C_{m_\alpha} \) is the most important longitudinal stability derivative and represents the static longitudinal stability of the vehicle. It is directly proportional to the distance, \( \Delta x \), by which the vehicle lies aft of the center of mass. For this reason, the said distance is called the longitudinal static margin. The longitudinal static margin is affected by the pitching moment contributions of the various components of the vehicle, such as wing, tail (or canard), fuselage, and nacelles. The primary contribution comes from the tail (or canard), where a change in the downwash (or upwash) is caused by the wing due to a change in the angle of attack. In addition, there can be significant changes in the static margin caused by the slipstream of a propeller. Clearly, for longitudinal static stability the vehicle must pitch in the negative direction (“downward” in pilot’s viewpoint), whenever the angle of attack increases (\( C_{m_\alpha} < 0 \)). A large majority of atmospheric vehicles have \( C_{m_\alpha} < 0 \), although some airplanes have been designed to be statically unstable from maneuverability considerations and require either exceptional piloting skills (Wright 1903 Flyer) or a closed-loop pitch stabilization system (the F-16 fighter) for maintaining equilibrium. The static stability also translates into the requirement that the vehicle must produce a positive pitching moment at \( \alpha = 0 \) for an ability to maintain equilibrium (\( C_m = 0 \)) at positive values of angle of attack (which is the normal situation for airplanes). Since an airplane’s wing normally produces a negative pitching moment in order to generate lift at \( \alpha = 0 \), a stable airplane needs a horizontal stabilizing surface (either a tail, or a canard) to provide the positive \( C_m \) at \( \alpha = 0 \).

The derivatives \( C_{x_\alpha} \) and \( C_{z_\alpha} \) largely represent the variation of drag and lift coefficients with angle of attack. Of these, \( C_{z_\alpha} \), approximately equaling the negative of lift–curve–slope (\( C_{L_{\alpha}} \)), is the more important and typically falls in the range of 4–6. The lifting effectiveness of the vehicle is measured by the magnitude of \( C_{z_\alpha} \).

The changes in aerodynamic force and moment do not occur instantaneously with the change in the angle of attack, but generally involve a time-lag due to the essentially circulatory flow over the lifting surfaces (Chapter 15).

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9 As defined for a lifting surface in Chapter 10, the aerodynamic center is the unique point about which the pitching moment is independent of the angle of attack. The concept of aerodynamic center can be extended for the whole vehicle, which may have several lifting surfaces. Usually, a vehicle’s aerodynamic center is called the neutral point, as it indicates the center of mass location for zero longitudinal static margin.
This aerodynamic time lag is referred to as *aerodynamic inertia* and is represented by the derivative $\dot{\alpha}$ and $\dot{\beta}$. Generally, $C_{x\alpha} \approx 0$, as thrust and drag are essentially noncirculatory in nature. On the other hand, the lag in lift and pitching moment can be large for a conventional airplane equipped with a horizontal stabilizer (tail) and have typical values of $C_{z\alpha} \approx -1$ and $C_{m\alpha} \approx -3$, respectively. We can derive $C_{z\alpha}$ from $C_{m\alpha}$ using the tail arm, $l_t$, as

$$C_{z\alpha} = \frac{\bar{c}}{l_t} C_{m\alpha}.$$ 

At hypersonic speeds encountered by re-entry vehicles, the aerodynamic lag is negligible, which results in all aerodynamic inertia derivatives approximated by zeros.

Finally, the stability derivatives $C_{x\alpha}$, $C_{z\alpha}$ and $C_{m\alpha}$ represent the effects of the pitch rate on lift and pitching moment. They are caused largely by the change in the angle of attack experienced by the lifting surfaces due to the curvature in the flight path. The derivative $C_{m\alpha}$ greatly influences the damping in the natural pitching oscillations and is thus known as *damping in pitch*. Generally, $C_{x\alpha} \approx 0$, while $C_{z\alpha}$ can be obtained by dividing $C_{m\alpha}$ by the nondimensional tail (or canard) arm, $\frac{\bar{c}}{l_t}$.

The kinematic attitude relations for the longitudinal motion are expressed as follows:

$$\dot{\Theta} = Q, \quad (13.146)$$

where $\Theta$, the disturbance in the pitch angle, is related to angle of attack and disturbance in the flight-path angle by

$$\alpha = \Theta - \phi. \quad (13.147)$$

From the last equation, it follows that $\Theta^c = \phi^c$, because we have employed the stability axes.

**Example 13.7.** A tail-less, delta-winged fighter airplane with $m = 10,455$ kg, $J_{yy} = 121,567$ kg.m$^2$, $\bar{c} = 6.95$ m, and $S = 60.5$ m$^2$, is undergoing a pitch-up maneuver with $A^c = 0$ and a constant Mach number, $M = 0.94$. The stability derivatives at the given Mach number are the following:

- $C_{m\alpha} = -0.31$/rad,
- $C_{m\alpha} = -1.44$/rad,
- $C_{m\alpha} = -1$/rad,
- $C_{z\alpha} = -2.85$/rad,
- $C_{z\alpha} = -2$/rad,
- $C_{z\alpha} = -1.39$/rad,
- $C_{z\alpha} = -0.37$,
- $C_{x\alpha} = -0.144$/rad,
- $C_{x\alpha} = -0.048$,
- $C_L = 0.146$. 


Simulate the ensuing motion of the airplane after reaching \( h^e = 2000 \) m and \( \delta^e = 45^\circ \), where an angle of attack disturbance is encountered with \( v = 0 \) and initial condition

(a) \( \phi^e = 0, Q^e = 0, \alpha = 0.01 \) rad.
(b) \( \phi^e = 0.1 \) rad, \( Q^e = 0.15 \) rad/s, \( \alpha = -0.1 \) rad.

**Table 13.9. M-file pitchup.m for Airplane’s Longitudinal State Equations**

```matlab
function xdot = pitchup(t,x)

% (c) 2006 Ashish Tewari
% Simulation due to gravity (oblate earth):
delta=x(6)
alt=x(1)
[g,gn]=gravity(alt+rm,delta);
% Atmospheric properties:
v=v0*(1+x(2));
atmosph=atmosphere(alt,v,c);
rho=atmosph(2);%density
q=0.5*rho*v^2;%dynamic pressure
mach=atmosph(3);
CL=m*g/(q*S);

%longitudinal dynamics:
phit=x(4)-x(3); %flight-path angle
hdot=v*sin(phit);
udot=-Q0+q*S*(Cxu*x(2)+Cxa*x(3)+
phi*(-cos(phi0)*CL+sin(phi0)*sin(A0)*m*gn/(q*S)))/(m*v0);
alphadot=(2*x(2)*omega*cos(delta)*sin(A0)+
2*x(2)*v0/m*(x(2)+Q0*x(5)+q*S*(Czu*x(2)+Cza*x(3))+
phi*(sin(phi0)*CL-cos(phi0)*cos(A0)*m*gn/(q*S)))+
c*Czq*x(5)/(2*v0)))/(1-q*S*c*Czad/(2*m*v0^2));

%theta dot:

Qdot=q*S*c*(Cma*x(3)+c*(Cmad*alphadot+Cmq*x(5))/(2*v0))/Jyy;
deltadot=v*cos(phit)*cos(A0)/(rm+alt);
xdot=[hdot;udot;alphadot;thetadot;Qdot;deltadot];
```

The simulation requires a numerical solution to the longitudinal dynamic and kinematic equations, with the prescribed initial condition. The aerodynamic force and moment are allowed to vary with a changing dynamic pressure in this simulation. The necessary computation is performed by the M-file `pitchup.m` tabulated in Table 13.9, which integrates the nonlinear differential equations of motion with the intrinsic MATLAB Runge–Kutta solver `ode45.m`. The results are plotted in Figs. 13.22–13.24. The departure from straight and level equilibrium condition [Case (a)] displays two distinct time scales of the airplane’s motion: a rapid and well-damped oscillation in the variables \( \alpha, \Theta, Q \) with settling time\(^{10}\) about 1 s and an insignificant change in speed and altitude.

\(^{10}\) In Chapter 14, *settling time* is defined as the time required for the response to decay to within ±2% of the steady state.
tude (Fig. 13.22), and a slower, less damped oscillation in altitude, speed, and pitch angle with a settling time of about 150 s with no appreciable variation in the angle of attack (Fig. 13.23). The clearly defined short- and long-period oscillations form the basis of the approximate longitudinal modes, as discussed ahead. The departure from a steady pitch-up maneuver [Case (b)] is plotted in Fig. 13.24. The steadily increasing altitude and a declining speed with a long-period oscillation in pitch are combined with a short-period pitching motion with variation in the angle of attack. If allowed uncorrected, the motion would quickly lead to the flight speed becoming zero, and then negative (called a tail slide).\footnote{A tail slide is normally avoided, as it causes destruction of trailing-edge control surfaces.} Note that the angle of attack remains small, thus the linear aerodynamic model remains valid, even though the speed falls to zero.

![Figure 13.22](image-url)

**Fig. 13.22.** The short-period response to angle of attack disturbance from straight and level flight.

**13.8.4 Airplane Longitudinal Modes**

The most common equilibrium condition encountered in an airplane is that of straight and level flight ($\Theta^e = Q^e = a^e_{xv} = a^e_{uv} = 0$). In such a condition,
a small disturbance caused by either the atmosphere or pilot input leads to two characteristic motions: (a) a long-period (or phugoid) oscillation in speed and altitude, in which the angle of attack remains constant, and (b) a rapid, short-period motion in which the angle of attack oscillates, but the speed remains unchanged. Approximate equations of motion for the phugoid and short-period modes can be easily derived from Eqs. (13.139)–(13.141), by making the relevant assumptions. In case of the phugoid oscillation, we neglect all variations with respect to the angle of attack and its time derivative, which amounts to disregarding the pitching motion caused by the change in the angle of attack, and taking $\alpha \approx 0$ in the remaining equations. Therefore, $\Theta \approx \phi$, and the resulting equations for phugoid approximation are the following, after neglecting the terms involving planetary rotation and curvature:

$$\frac{m v_e}{qS} \ddot{u} = C_{x_u} u - \Theta \frac{mg_e}{qS}. \tag{13.148}$$

$$\frac{m v_e}{qS} \dot{Q} = C_{z_u} u + \Theta \cos \psi_e \frac{mg_e}{qS} + \frac{c}{2v_e} C_{z_Q} Q. \tag{13.149}$$
On substituting Eqs. (13.146) and (13.149) into (13.148), we have
\[
\begin{align*}
\left[ \left( \frac{mv^e}{qS} \right)^2 + \frac{1}{C_{z_u}} + \frac{\bar{c}}{2qSC_{z_u}} \right] \dot{\Theta} + \left[ \frac{mv^e a}{qSC_{z_u}} - \frac{C_{x_u}}{C_{z_u}} \left( \frac{mv^e}{qS} + \frac{\bar{c} e}{2v^e C_{z_u}} \right) \right] \Theta \\
- qS \left( C_L + \frac{C_{x_u}}{C_{z_u}} \right) \Theta = 0,
\end{align*}
\]
(13.150)
\[
u = - \left( \frac{mv^e}{qS} + \frac{\bar{c} e}{2v^e C_{z_u}} \right) \frac{\dot{\Theta}}{C_{z_u}} - \frac{a}{C_{z_u}} \Theta,
\]
(13.151)
where
\[
C_L = \frac{mg^e}{qS}
\]
(13.152)
is the equilibrium lift coefficient and
\[
a = \cos \Psi \frac{mg^e}{qS}.
\]
(13.153)
Taking the Laplace transform of Eq. (13.150), we can write the characteristic equation for the second-order system (Chapter 14) as follows:
\[
s^2 + 2\zeta \omega_s + \omega^2 = 0,
\]
(13.154)
where the natural frequency, $\omega$, of the phugoid mode is given by

$$\omega^2 = -C_{za}(C_L + a C_{za}) \left( \frac{m v_e q S}{q S} \right)^2 + \frac{m c_{qa}^2}{2 q S} C_{za},$$

(13.155)

and the phugoid damping ratio, $\zeta$, is

$$\zeta = \frac{\frac{m v_e q S}{q S} - C_{za}\left( \frac{m v_e q S}{q S} + \frac{q}{q S} C_{za} \right)}{2 \omega \left( \frac{m v_e q S}{q S} \right)^2 + \frac{m c_{qa}^2}{2 q S} C_{za}}.$$

(13.156)

Thus, the phugoid mode is a pure pitching motion, whose frequency and damping depend on the lift coefficient, mass, and stability derivatives $C_{xa}, C_{za}, C_{qa}$. Usually, the phugoid frequency and damping are both quite small, representing an almost-constant amplitude, long-period oscillation. The approximately conservarive flight path (Chapter 12) indicates a slow exchange between the potential and kinetic energies.

**Example 13.8.** Find the phugoid characteristics (frequency and damping) of a jet transport airplane [46] with $m = 84,891$ kg, $J_{yy} = 3,564,403$ kg.m$^2$, $e = 6.16$ m, $S = 223$ m$^2$, $l_t = 17.8$ m, flying straight and level at 12.2 km altitude, $\Psi_e = 30^\circ$, and $M = 0.62$, where $v_e = 182$ m/s, $q = 5036.79$ N/m$^2$, and

$C_{m,\alpha} = -0.619$/rad,

$C_{m,\theta} = -11.4$/rad,

$C_{m,\phi} = -3.27$/rad,

$C_{za} = -4.46$/rad,

$C_{za} = C_{m,\alpha} \frac{e}{l_t} = -3.94$/rad,

$C_{za} = C_{m,\alpha} \frac{e}{l_t} = -1.13$/rad,

$C_{za} = -1.48,$

$C_{za} = 0.392$/rad,

$C_{za} = -0.088,$

$C_L = 0.74.$

Substituting these values into Eqs. (13.155) and (13.156), we get the phugoid frequency and damping ratio to be the following:

$\omega = 0.07627$ rad/s,

$\zeta = 0.04215$,

which results in the time period of $T = \frac{2\pi}{\omega} = 82.38$ s and a settling time (Chapter 14) of $t_s = \frac{\zeta \omega}{\zeta \omega} = 1244$ s.
The short-period longitudinal mode for an airplane is represented by neglecting the variation in the forward speed \((u = \dot{u} = 0)\) from a straight and level equilibrium flight, as well as the effects of planetary rotation and curvature. Furthermore, for an airplane the term involving oblate gravitation, \(g^g_{\delta}\), is usually ignored, resulting in a gravity-free dynamical model, given by

\[
\frac{mv^e}{qS}(\dot{\alpha} - Q) = C_{z_a} \alpha + \frac{\bar{e}}{2v^e}(C_{z_a} \dot{\alpha} + C_{z_q} Q), \tag{13.157}
\]

\[
\frac{J_{yy}}{qS\bar{c}} \dot{Q} = C_{m_\alpha} \alpha + \frac{\bar{e}}{2v^e}(C_{m_\alpha} \dot{\alpha} + C_{m_\theta} \dot{\theta}). \tag{13.158}
\]

Here, the equation for forward translation has been discarded. The short-period mode is thus a two-degree-of-freedom motion involving pitch and vertical translation (plunge). The predominant stability derivatives in the short-period mode are \(C_{m_\alpha}, C_{z_a}, C_{m_q}, C_{z_q}\). A state-space representation (Chapter 14) of the short-period dynamics is written as follows:

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{\theta} \\
\dot{Q}
\end{bmatrix} =
\begin{bmatrix}
\frac{C_{z_a}}{\Delta} & 0 & \frac{mv^e + \bar{e}C_{z_a}}{2v^e} \\
0 & 0 & 1 \\
\frac{qS^2}{J_{yy}}(C_{m_\alpha} + \frac{\bar{e} - C_{m_\alpha} C_{z_a}}{2}) & 0 & \frac{qS^2}{4v^e J_{yy}}(C_{m_q} + C_{m_\theta} \frac{mv^e + \bar{e}C_{z_q}}{\Delta})
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\theta \\
Q
\end{bmatrix}, \tag{13.159}
\]

where

\[
\Delta = \frac{mv^e}{qS} - \frac{\bar{e}C_{z_a}}{2v^e}.
\]

**Example 13.9.** Find the longitudinal short-period characteristics (frequency and damping) of the jet transport airplane of Example 13.8.

The necessary computations are performed with the use of Eq. (13.159) and the following MATLAB statements (assuming all the necessary constants have been defined in the workspace):

```matlab
def mavv/(q*S)-c*Czad/(2*v);
A = [Cza/D 0 (mv^e/(q*S)+c*Czq/(2*v))/D;
0 0 1;
(q*S^2/Jyy)*(Cma+c*Chad*Cza/(2*v*D)) 0...
(q*S^2/(2*v*Jyy))*(Cma+c*Chad+mv^e/(q*S)+c*Czq/(2*v))/D]
A = -0.3238 0 0.9938
0 0 1.0000
-1.1668 0 -0.4812
>> damp(A)
```

**Example 13.9.** Find the longitudinal short-period characteristics (frequency and damping) of the jet transport airplane of Example 13.8.
Thus, the short-period natural frequency and damping-ratio are given by
\[ \omega = 1.15 \text{ rad/s}, \]
\[ \zeta = 0.351, \]
which results in the time period of \( T = \frac{2\pi}{\omega} = 5.46 \text{ s} \) and a settling time of \( t_s = \frac{1}{\zeta \omega} = 9.91 \text{ s} \).

The analysis presented above is that of the controls-fixed case, where no activation of the aerodynamic control takes place. This condition is rarely met in practice as the control surfaces are not rigidly attached to the vehicle, and thus undergo some deflection whenever the relative flow changes due to the vehicle's motion. The aerodynamic control surface for longitudinal dynamics is the elevator. The elevator can take various forms, such as the conventional trailing-edge device on the horizontal tail, the all-moving stabilizer, and the elevons of a tail-less aircraft. The vertical force and pitching moment contributions of the elevator are assumed linear and are modeled by the stability derivatives \( C_z, C_m \). Often it is necessary to model the actuating mechanism of the elevator using a second-order dynamical system called the actuator. In addition, it is also necessary to model the airplane’s motion with a free elevator, where the surface is completely free to assume an equilibrium position depending upon the angle of attack it experiences. Needless to say, the ideal controls-free condition is rarely met in practise due to inertia, friction, and stiffness of the actuating mechanism. Since the controls-free condition is also the case of zero control force (or moment) exerted by the actuating mechanism, it is desirable to control the free-elevator deflection, often using a smaller trailing-edge surface called a trim tab. A pilot uses a trim tab whenever small pitch adjustments are required, and is very useful in relieving the control force. Hence, a trim tab is somewhat like the momentum wheel in a spacecraft, where small attitude changes can be made without direct external force of the rocket thrusters. A variation of the trim tab is the servo-tab, wherein the deflection of an all-moving stabilizing surface is controlled by a gearing mechanism.

### 13.8.5 Lateral Dynamics

The lateral dynamic equations—Eqs. (13.127), (13.130), and (13.132)—involve three degrees of freedom, namely translation along \( oy \) (sideslip), and rotation about \( ox \) (roll) and \( oz \) (yaw). In a nondimensional form, with no longitudinal coupling (\( \phi = \alpha = \epsilon = Q = v = 0 \)), these equations can be written as follows:
\[ \begin{align*}
m(v^e \dot{\beta} - \frac{v^e}{r^e} \cos^2 \Theta^e \tan \delta^e \cos \Psi^e \dot{\beta} \\
-2\Omega v^e \dot{\beta} \cos \delta^e \sin \Theta^e \sin \Psi^e + v^e R) \\
= mg_\epsilon^e \Phi \cos \Phi^e \cos \Theta^e + mg_\epsilon^e \Phi \cos \Phi^e \sin \Theta^e \sin \Psi^e + \Phi \sin \Phi^e \sin \Psi^e - \Psi \cos \Phi^e \cos \Psi^e \\
+ qS[C_{y\beta} \dot{\beta} + \frac{b}{2\delta^e}(C_{y\beta} \dot{\beta} + C_{y\beta} P + C_{y\beta} R)]. \\
J_{xz}\dot{P} - J_{zz}\dot{R} + (J_{zz} - J_{yy}) RQ^e \\
= qSb[C_{\beta\beta} \dot{\beta} + \frac{b}{2\delta^e}(C_{\beta\beta} \dot{\beta} + C_{\beta\beta} P + C_{\beta\beta} R)]. \\
J_{zz}\dot{R} - J_{xz}\dot{P} + J_{xz} RQ^e + (J_{yy} - J_{xx}) PQ^e \\
= qSb[C_{\beta\beta} \dot{\beta} + \frac{b}{2\delta^e}(C_{\beta\beta} \dot{\beta} + C_{\beta\beta} P + C_{\beta\beta} R)].
\end{align*} \]

The lateral motion thus consists of a translational (sideslip) and two rotational (roll and yaw) degrees of freedom. The lateral stability derivatives are defined in the same manner as that of the longitudinal derivatives, except that the characteristic length is the wing span, \( b \). The derivatives \( C_{y\beta} \) (sidewise force due to steady sideslip), \( C_{n\beta} \) (static directional stability), \( C_{l\beta} \) (dihedral effect), \( C_{n\alpha} \) (damping in yaw), and \( C_{l\alpha} \) (damping in roll) are the predominant lateral stability derivatives. While the sideslip and yaw rate derivatives are influenced by the fuselage and the vertical tail (fin), the roll-rate derivatives are mainly due to the wing. The static directional stability, \( C_{n\beta} \), determines the ability of the aircraft to regain its equilibrium heading, \( \Psi^e \), once displaced by a sideslip. Also known as weathercock stability, \( C_{n\beta} > 0 \) is required for directional stability and increases with the nondimensional product (called fin volume ratio) of the fin arm, \( l_f \), and fin area, \( S_f \). While the predominant contribution to directional stability comes from the fuselage, fin, and nacelles, a swept wing can have a significant stabilizing influence. The dihedral effect, \( C_{l\beta} \), is caused by the effective dihedral angle of the wing, as well as the lift produced by the fin. Due to the positive dihedral angle, a negative value of \( C_{l\beta} \) is created, which turns the aircraft by banking it in the direction opposite to the sideslip. The damping in yaw is caused by the change in the fin’s angle of attack due to a yaw rate, which tends to apply an opposite yawing moment (\( C_{n\alpha} < 0 \)). The damping in roll is due to the change in the wing angle of attack due to a roll rate and is such that an opposing rolling moment is created (\( C_{l\alpha} < 0 \)). The derivative \( C_{n\alpha} \) is due to the same effect as \( C_{l\alpha} \), which causes a differential drag on the two wings, thereby generating an adverse yawing moment (\( C_{n\alpha} < 0 \)). The other roll-yaw coupling derivative is \( C_{l\alpha} \), which is due to a differential lift on the two wings due to a yaw rate. The other rate derivatives (\( C_{y\beta}, C_{y\beta}, C_{y\beta}, C_{y\beta} \)) are usually negligible and are ignored in a stability analysis.

The rudder is the aerodynamic control surface for yaw and sideslip, while the rolling motion is controlled by the ailerons. The rudder is a trailing-edge
control surface on the fin and acts like the elevator, while the ailerons are mounted on the wing trailing edges and are differentially deployed to create a rolling moment. The control derivatives, $C_{m_a}$, $C_{l_a}$, $C_{n_a}$, $C_{l_r}$, $C_{b_y}$, model the linear effects of deploying the aileron and rudder by the respective deflection angles, $\delta_a$ and $\delta_r$.

All lateral stability derivatives are strong functions of the Mach number, and some of them can change sign while crossing through the transonic regime. This is especially true for the yawing moment derivatives. Due to reduced static directional stability and yaw damping at supersonic speeds, most supersonic aircraft require a yaw stability augmentation system with a closed-loop activation of rudder.

The kinematic attitude relations for the small-disturbance lateral motion can be expressed in terms of the 3-2-1 Euler angles as follows (Chapter 2):

$$\dot{\phi} = P + R \tan \Theta^e,$$
$$\dot{\psi} = R \sec \Theta^e.$$  

Clearly (as pointed out in Chapter 2), $\Theta^e = \pm 90^\circ$ is a point of singularity for this attitude representation.

**Example 13.10.** Simulate the lateral response of an axisymmetric, ballistic reentry vehicle with the following parameters to be an initial sideslip and roll disturbance of $\beta_0 = 0.01$ rad and $P_0 = -0.001$ rad/s; $m = 92$ kg, $J_{xx} = 0.972$ kg.m$^2$, $J_{yy} = J_{zz} = 9.32$ kg.m$^2$, base radius, $b = 0.22$ m, and base area, $S = 0.152$ m$^2$. The stability derivatives of the vehicle based upon the base area and base radius at the equilibrium flight condition of $V^e = 5000$ m/s, $h^e = 25$ km, $\delta^e = 45^\circ$, $\Psi^e = 190^\circ$, and $\Theta^e = -85^\circ$ are the following:

$$C_{m_a} = -C_{n_a} = -0.52/\text{rad},$$
$$C_{m_q} = C_{n_q} = -8/\text{rad},$$
$$C_{z_a} = C_{y_a} = -2.15/\text{rad},$$
$$C_{z_q} = C_{y_q} = -0.35/\text{rad},$$
$$C_{x_a} = -2C_D = -0.2,$$
$$C_{l_p} = -0.002/\text{rad},$$
$$C_D = 0.1.$$

The other stability derivatives vanish due to the absence of lifting surfaces and the hypersonic speed.

In order for us to perform the simulation, a program named `lateralentry.m`, which is tabulated in Table 13.10, is written to integrate the equations of lateral motion with the intrinsic MATLAB Runge–Kutta solver `ode45.m`. The results are plotted in Figs. 13.25 and 13.26. The sideslip, roll, and yaw angles, as well as the yaw rate, are seen to oscillate in Fig. 13.25 with a decreasing amplitude for the 2 s of simulation, while the roll rate remains nearly constant in the given duration. The decay of roll rate is much slower than the yaw.
13.8.6 Airplane Lateral Modes

As in the longitudinal dynamics, the lateral motion of an airplane can be represented by a combination of distinct modes, each of which is obtained as
an approximation of the actual equations of motion. The common airplane
equilibrium of straight and level flight ($\Theta^e = Q^e = a^e_{xv} = a^e_{yv} = a^e_{zv} = \Phi^e =
\n^e = R^e = 0$) is the starting point for the lateral modes, the simplest of
which is the pure rolling motion created by an aileron input, called the roll-
subsidence mode, and described by the following rolling moment equation:

$$ J_{xx} \dot{P} = \frac{b}{2v\epsilon} C_{lp} P, \quad (13.165) $$

where $P = \dot{\Phi}$. Clearly, the mode is a first-order dynamical system with an
exponentially decaying response,

$$ P(t) = P(0)e^{\frac{2v\epsilon}{J_{xx}} J_{xx} \dot{P}}. \quad (13.166) $$

For a given speed and altitude, the rapidity with which the roll rate goes
to zero is primarily dependent upon the ratio $\frac{J_{xx}}{2v\epsilon. C_{lp}}$. However, since $J_{xx}$ is
roughly proportional to the square of the wing span, the rate of decay of roll
rate is primarily determined by the magnitude of $C_{lp}$. The bank angle can be
obtained by integrating Eq. (13.166) as follows:

$$ \Phi(t) = \int P(t)dt = \Phi(0) + P(0) \frac{2v\epsilon J_{xx}}{Q_{Sb^2 C_{lp}}} e^{\frac{2v\epsilon}{J_{xx} \dot{P}}} C_{lp} t. \quad (13.167) $$

**Fig. 13.25.** Lateral dynamic response of a re-entry vehicle.
Another lateral approximation is the short-period Dutch-roll mode, where the sideslip and yaw are coupled. Such a motion is normally generated by a rudder input. By assuming a negligible rolling motion, the airplane’s attitude remains nearly wings’ level, and we have $\beta \approx -\Psi$. Hence, the rolling moment and sideslip equations are discarded, and the Dutch-roll dynamics is given by the following yawing moment equation:

$$\frac{J_{zz}}{qSb} \ddot{\beta} - \frac{b}{2\nu e} C_{n_r} \dot{\beta} + C_{n_\beta} \beta = 0. \quad (13.168)$$

Clearly, the frequency and damping of the Dutch-roll mode are the following:

$$\omega = \sqrt{\frac{qSbC_{n_\beta}}{J_{zz}}}, \quad (13.169)$$

$$\zeta = -C_{n_r} \sqrt{\frac{qSb^3}{32J_{zz}C_{n_\beta}}}. \quad (13.170)$$

Thus, the damping in the Dutch-roll is directly proportional to $C_{n_r}$, while its frequency is determined by the ratio $\frac{C_{n_\beta}}{C_{n_r}}$. The assumption of negligible roll in the traditional Dutch-roll approximation is valid only if the dihedral effect, $C_{l_d}$, is reasonably small in magnitude. For an airplane with a significantly large
magnitude of $C_{lb}$, the Dutch-roll mode includes a distinct rolling motion with a reduced damping ratio and can be uncomfortable for passengers as well as bad for weapons-aiming purposes.

A third lateral mode is the *spiral divergence*, which consists of an ever-increasing bank angle, coupled with the yaw angle. The flight path is a slowly steepening coordinated turn. The equations of motion for the spiral mode can be obtained by neglecting the sideslip equation, and putting $\beta = 0$ in the roll and yaw equations:

\[
J_{xx} \dot{\Phi} - J_{xz} \dot{\Psi} = \frac{q S b^2}{2 \nu e} \left( C_{lp} \Phi + C_{lr} \Psi \right),
\]

\[
J_{zz} \dot{\Psi} - J_{xz} \dot{\Phi} = \frac{q S b^2}{2 \nu e} \left( C_{np} \Phi + C_{nr} \Psi \right).
\]

The coupled roll-yaw motion leads to a single-degree-of-freedom, first-order dynamical system with a real, positive eigenvalue for the usually unstable spiral mode. However, the eigenvalue is generally small in magnitude, leading to a large time constant. Due to its long-period characteristic, the slowly diverging spiral is easily compensated for by the pilot. The requirement of stability in the spiral mode can be obtained by examining the constant term in the lateral characteristic equation [45],

\[
C_{lb} C_{nr} - C_{nr} C_{lr},
\]

which must be positive for the real root to lie in the left-half Laplace plane (stable spiral mode). Hence, for spiral stability we require $C_{lb} C_{nr} > C_{nr} C_{lr}$. The yawing moment derivatives, $C_{nr}, C_{nl}$ are similarly affected by the size of the fin, hence increasing one also results in the increase of the other. The cross-derivative $C_{lr}$ is primarily dependent upon the lift coefficient, and thus cannot be arbitrarily selected at a given speed–altitude combination. This leaves only the dihedral effect, $C_{lb}$, as the design parameter, which can be selected through a proper wing dihedral angle. However, increasing the magnitude of the dihedral effect—while leading to a greater spiral stability—causes the rolling moment in the Dutch-roll motion to become large, resulting in a reduced damping in the coupled roll-yaw-sideslip dynamics. Since damping of the short-period Dutch-roll motion must remain adequate in most airplanes, a small amount of long-period spiral instability is accepted as a compromise in the design.

### 13.8.7 Rotational Motion of a Launch Vehicle

The atmospheric trajectory of a launch vehicle (or a ballistic missile) may involve an appreciable aerodynamic torque due to the presence of stabilizing fins. As discussed in Chapter 12, it is crucial for a launch vehicle to be maintained at a zero angle of attack and sideslip due to aerodynamic load
considerations. The attitude control of launch vehicles primarily involves control torque, $M_c = M_{cx} \mathbf{i} + M_{cy} \mathbf{j} + M_{cz} \mathbf{k}$, produced by thrust vectoring. Euler’s equations of motion for a launch vehicle with pitch-yaw symmetry ($J_{yy} = J_{zz}$) can thus be written as follows:

\[
J_{xx} \dot{P} = M_{cx} + qSb \frac{b}{2v} C_{l_p},
\]

\[
J_{yy} \dot{Q} + PR(J_{xx} - J_{yy}) = M_{cy} + qSb \left[ \frac{b}{2v} (C_{m_q} Q + C_{m_a} \alpha) + C_{m_a} \alpha \right],
\]

\[
J_{yy} \dot{R} + PQ(J_{yy} - J_{xx}) = M_{cz} + qSb \left[ \frac{b}{2v} (C_{n_r} R + C_{n_p} \beta) + C_{n_p} \beta \right],
\]

where $b$ refers to the maximum fin span, or the maximum body diameter in case of a vehicle without fins. Due to axisymmetry, the stability derivatives due to pitch and yaw are indistinguishable from one another. Thus, we have $C_{z_\alpha} = C_{y_\beta}$, $C_{m_q} = C_{n_r}$, $C_{m_a} = -C_{n_p}$, and $C_{m_a} = -C_{n_q}$. For most launch vehicles and ballistic missiles, the stability derivatives representing aerodynamic lag in lift and side force are negligible, because of the small size of the lifting surfaces. Thus, we can assume $C_{z_\alpha} = C_{y_\beta} \approx 0$. Generally, the roll rate, $P$, is quite small; hence, no appreciable force is created by the Magnus effect [22].

The control torque components $M_{cy}$ and $M_{cz}$ are created by the small thrust deflection angles, $\epsilon$ and $\mu$ (Fig. 13.27),

\[
M_{cy} = l_x f_T \epsilon,
\]

\[
M_{cz} = -l_x f_T \mu,
\]

where $l_x$ is the longitudinal distance of the nozzle from the center of mass, and $f_T$ denotes the thrust. The rolling control torque, $M_{cx}$, is generated aerodynamically through control surface deflection, $\delta$, and is given by the linear relationship

\[
M_{cx} = C_{l_\delta} \delta.
\]

The control by thrust vectoring produces undesirable lift and side force if there are no opposing aerodynamic force components generated by the body and fins. This is the case during initial lift-off when the airspeed is too small for the opposing aerodynamic force to be created. In such a situation, the lateral support is provided to the vehicle either by the launch tower or by rocket thrusters near the nose of the vehicle. We shall make the assumption that lateral translation of the vehicle is prevented by such a mechanism; therefore, it is not necessary to consider the translational motion when the aerodynamic force and moment are negligible. As the speed increases within a few seconds after launch, the aerodynamic force and moment become sufficiently large for the vehicle to be treated in a manner similar to an aircraft, but with additional pitch-yaw symmetry. It is to be noted that due to the continuously active attitude stabilization system, the angle of attack and sideslip are kept small; thus, the assumption of small disturbances is more valid for the launch
vehicle than the airplane. Hence, we can confidently utilize the results of the linearized longitudinal translation model presented earlier and write the vehicle’s angle of attack (and sideslip) dynamical equations by the short-period approximation relative to a spherical gravity model as follows:

\[
\frac{mv}{qS} \dot{\alpha} = \frac{mv}{qS} Q - \Theta \frac{mg}{qS} \sin \Theta e + C_{z\alpha} \alpha + \frac{f_T \epsilon}{qS} \tag{13.177}
\]

and

\[
\frac{mv}{qS} \dot{\beta} = -\frac{mv}{qS} R + \frac{mg}{qS} \Psi \sin \Theta e + C_{y\beta} \beta + \frac{f_T \mu}{qS}. \tag{13.178}
\]

Note that we have adopted the Euler angle representation, \( (\Psi)_3, (\Theta)_2, (\Phi)_1 \), for the vehicle’s attitude relative to the local horizon, which is nonsingular as long as \( \Theta e \neq \pm 90^\circ \). This is acceptable, because the vertical pitch angle occurs only at lift-off, which lies outside our domain of analysis (due to zero aerodynamic force and moment at that point). Furthermore, the angles of attack and sideslip are assumed negligible in comparison with the pitch and yaw angles in the gravity terms.

Example 13.11. Consider the Vanguard ballistic missile with the following parameters [46] at flight condition of maximum dynamic pressure, \( q = 28,035 \text{ N/m}^2 \), which occurs 75 s after launch at relative speed, \( v = 392 \text{ m/s} \), standard altitude 11 km, and mass \( m = 6513.2 \text{ kg} \):
\[ b = 1.1433 \text{ m}, \]
\[ S = 1.0262 \text{ m}^2, \]
\[ l_x = 8.2317 \text{ m}, \]
\[ C_{l_p} = C_{m_a} = C_{l_i} = 0, \]
\[ \Theta^e = 68.5^\circ, \]
\[ C_{z_a} = -3.13 \text{ /rad}, \]
\[ C_{m_a} = 11.27 \text{ /rad}, \]
\[ f_T = 133, 202.86 \text{ N}, \]
\[ J_{yy} = 156, 452.8 \text{ kg.m}^2, \]
\[ \frac{b}{2v} C_{m_q} = -0.321 \text{ s}. \]  

(13.179)

Since the missile is not equipped with fins and aerodynamic control surfaces, we do not require roll control; therefore, the first Euler equation yields a missile spinning at a constant rate, which we can take to be zero without any loss of generality. For \( P = 0 \), the dynamic and kinematic equations of motion of the missile can be represented by the following set of linear differential equations (state equation of Chapter 14) at the given flight point:

\[
\begin{bmatrix}
\dot{Q} \\
\dot{R} \\
\dot{\theta} \\
\dot{\psi} \\
\dot{\alpha} \\
\dot{\beta}
\end{bmatrix} =
\begin{bmatrix}
gSb^2 C_m q \\
0 \\
0 \\
1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
-\frac{1}{\cos \Theta^e} \\
-\frac{q \sin \Theta^e}{v} \\
-\frac{q \sin \Theta^e}{v}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
-\frac{1}{\cos \Theta^e} \\
-\frac{q \sin \Theta^e}{v} \\
-\frac{q \sin \Theta^e}{v}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
-\frac{1}{\cos \Theta^e} \\
-\frac{q \sin \Theta^e}{v} \\
-\frac{q \sin \Theta^e}{v}
\end{bmatrix}
\begin{bmatrix}
\frac{l_x f_T}{J_{yy}} \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\epsilon \\
\mu
\end{bmatrix}.
\]

\[ \text{The axisymmetric missile without fins would remain nonrolling after being launched at a zero roll rate. Conversely, a rifle bullet continues rolling at a fixed rate after leaving the barrel.} \]
Let us simulate the vehicle’s response for 0.1 s due to a step change in both $\epsilon$ and $\mu$ by $1^\circ$ at the given flight point using the Simulink block diagram shown in Fig. 13.28. The exponentially increasing response, Fig. 13.29, indicates an unstable equilibrium, which requires an automatic attitude stabilization system. The instability is caused by a positive value of $C_{m_{\alpha}} (= -C_{n_{\beta}})$, as well as the effect of gravity on the angle of attack (and sideslip) due to $\theta^e \neq 0$. If large fins are added near the aft part of the vehicle, it could be made statically stable like an airplane and some surface-to-air missiles. The response shown in Fig. 13.29 would differ appreciably from that of the actual vehicle, because we have not modeled structural flexibility and fuel-slosh dynamics.

![Simulink diagram](image_url)

**Fig. 13.28.** Simulink diagram for step response of the Vanguard missile without attitude stabilization.

### 13.8.8 Inertia Coupled Dynamics

The separation of atmospheric rotational dynamics into longitudinal and lateral motions involves the assumption of small angular rate disturbances. For some vehicles, such as modern fighter airplanes and missiles, the concentration of mass in the fuselage results in a large difference between the rolling moment of inertia and the pitching (and yawing) inertia. Consequently, the nonlinear coupling terms—such as $(J_{xx} - J_{zz})PR$ in the pitching moment equation—become significant, thereby causing an interaction between the longitudinal and lateral dynamics for even moderate body rates. Inertial coupling has caused some fighter airplanes (e.g., the *Lockheed F-104*) to become unstable when rolling at high rates. While an approximate linear stability analysis is possible by considering the coupled longitudinal short-period and Dutch-roll dynamics [46], it is often necessary to simulate the complete six-degree-of-freedom dynamics of an inertia coupled vehicle, as discussed in Chapter 15. Controlling the inertia coupled dynamics usually requires a multivariable control system (Chapter 14). The case of aerodynamic missiles and artillery shells rolling at high rates is further aggravated by the aerodynamic coupling caused by the *Magnus effect* [22] and generally results in additional stability derivatives in a linearized stability analysis [45]. The aerodynamic behavior of a fighter aircraft rolling at a large angle of attack is essentially
13.8 Attitude Motion in Atmospheric Flight

Fig. 13.29. Step response of the Vanguard missile without attitude stabilization.

nonlinear and leads to a complex motion (Chapter 15). Here, we neglect the aerodynamic coupling effects and confine our attention to the inertia coupled dynamics. Euler’s equations of rotational dynamics of an inertia coupled vehicle, Eq. (13.19), can be expressed as follows:

\[
\begin{align*}
\dot{p} &= \frac{J_{zz}(J_{xx} + J_{yy})PQ - [J_{zz}^2 + J_{zz}(J_{zz} - J_{yy})]QR + J_{zz}N + J_{zz}L}{J_{xx}J_{zz} - J_{xx}^2}, \\
\dot{q} &= \frac{J_{xx}(R^2 - P^2) + (J_{zz} - J_{zz})PR + M}{J_{yy}}, \\
\dot{r} &= \frac{J_{zz}(\dot{P} - QR) + (J_{xx} - J_{yy})PQ + N}{J_{zz}},
\end{align*}
\]

(13.180)

where \(L, M, N\) are the external torque components acting on the vehicle. In addition to the rotational dynamic equations, the linear equations of small-disturbance aerodynamic translation, \(u, \alpha, \beta\) [Eqs. (13.126)–(13.128)], are required for propagating the aerodynamic force and moment in time. Finally, the attitude kinematics are represented by a suitable representation, such as 3-2-1 Euler angles, or the quaternion.

Example 13.12. Simulate the inertia-coupled response of the fighter airplane to an initial roll-rate disturbance of 0.5 rad/s when flying straight and level at standard sea level and \(\delta = 45^\circ\), with \(\theta = 45^\circ\) and Mach number, \(M = 0.797\).
The lateral dynamic data of the airplane at the prescribed Mach number are given in Exercise 13.14, while its longitudinal characteristics are given by 

\( J_{yy} = 36,110.67 \text{ kg.m}^2, \bar{c} = 1.95 \text{ m}, t = 5.64 \text{ m}, \) and the following stability derivatives:

\[
\begin{align*}
C_{m,\alpha} &= -0.44/\text{rad}, \\
\frac{\bar{c}}{2vE}C_{m,q} &= -0.0305/\text{rad}, \\
\frac{\bar{c}}{2vE}C_{m,\alpha} &= -0.0159/\text{rad}, \\
C_{z,\alpha} &= -5.287/\text{rad}, \\
\frac{\bar{c}}{2vE}C_{z,q} &= -0.01055/\text{rad}, \\
\frac{\bar{c}}{2vE^2}C_{z,\alpha} &= -0.0055/\text{rad}, \\
C_{z,u} &= -0.185, \\
C_{x,\alpha} &= 0/\text{rad}, \\
C_{x,u} &= -0.0426.
\end{align*}
\]

The six-degree-of-freedom simulation is performed with the coupled equations of motion encoded in \textit{aircoupled.m} (Table 13.11). The response of the aircraft is plotted in Figs. 13.30–13.32. The observed response to the large roll rate can be broken into three distinct phases:

(a) \(0 < t \leq 10 \text{ s},\) during which the roll rate, yaw rate, and sideslip angle undergo a rapidly decaying oscillation. During this time, the speed, pitch angle, pitch rate, and angle of attack are unchanged, while the bank and yaw angles increase slowly with time.

(b) \(10 < t \leq 170 \text{ s},\) in which all variables except the roll rate undergo an unstable long-period oscillation and reach their maximum values near the end of the interval. This interval represents a diving attitude, with ever-increasing peak speed and increasingly negative angle of attack, which leads to a supersonic Mach number and more than double the equilibrium speed at \(t = 110 \text{ s}.\)

The attitude angles also build up in this phase, with the pitch angle reaching a maximum magnitude of \(\Theta = -220^\circ\) at \(t = 108 \text{ s},\) which is a case of inverted flight. By this time, a steep bank and dive are established. The main reason for the increase in the pitch rate during this interval is the small damping in pitch, \(C_{m,\alpha},\) at the given Mach number.

(c) \(t > 170 \text{ s},\) which sees the airplane trying to recover from the unusual pitch attitude in stable pitching oscillations that converge to a steady pitch angle of \(\Theta = -90^\circ.\) In this phase, rolling and yawing motions increase exponentially, leading to an ever-steepening, downward spiral. This is the classical spiral divergence discussed above. There is a negligible variation in the speed and angle of attack in this phase, while the sideslip angle shows a steep rise in magnitude.
It is important to note that the flow angles $\alpha, \beta$ remain small during this nonlinear simulation, thereby validating the assumption of linearized aerodynamics. Hence, the large body rates and angles are not caused by large aerodynamic disturbances, but rather by inertia coupling. The increase in the speed during the steep dive resulting from the high roll-rate disturbance leads to the airplane’s entering the supersonic regime. We have not accounted for the variation of the stability derivatives at transonic and supersonic Mach numbers in this simulation, which may cause the aerodynamic torque to be appreciably modified in the last phase. A common condition in transonic...
flight is the *tuck-under* phenomenon, wherein the aerodynamic center moves aft, and the damping derivatives diminish in magnitude as the Mach number increases.

Another situation where inertia coupling becomes important is when the angular momentum of rotors in aircraft engines is taken into account. In such situations, the modeling of inertia coupling is carried out in the same manner as that given above for spacecraft with rotors, by adding the constant angular momenta of the spinning rotors to Euler’s equations.

### 13.9 Summary

*Euler’s equations of rotational motion* govern the rotational dynamics of rigid bodies, and their solution gives the angular velocity at a given instant. Along with the kinematic equations, Euler’s equations completely describe the changing attitude of a rigid body under the influence of a time-varying torque vector. When expressed in a body-fixed frame, Euler’s equations involve constant moments and products of inertia. In a principal body-fixed frame, the products of inertia vanish, yielding a diagonal inertia tensor. Torque-free motion of rigid spacecraft is an example of conservative rotational maneuvers,
wherein both angular momentum and rotational kinetic energy are conserved. While a rigid spacecraft’s rotation about either the minor or the major axis is unconditionally stable, a semirigid spacecraft always tends toward the state of equilibrium with the lowest rotational kinetic energy—a pure spin about the major axis. Time-optimal maneuvers are an important open-loop method of controlling the spin and attitude of spacecraft and consist of at least a pair of suitably timed, equal and opposite torque impulses (bang-bang control). Other methods of stabilizing and controlling spacecraft’s attitude motion are the use of rotors (dual-spin, reaction/momentum wheels, and control moment gyroscope), gravity gradient and magnetic torques. When considering the rotational dynamics within the atmosphere, Euler’s equations are employed with the assumption of a rigid vehicle, and taking into account the aerodynamic torque generated by the rotation of the vehicle, as well as the control torque applied either by the pilot, or by an automatic control system. The airplane is the generic atmospheric flight vehicle for attitude motion models. The Coriolis acceleration terms due to planetary rotation and flight-path curvature are generally negligible in a rotational model, except for that of an atmospheric entry vehicle. The linearized aerodynamic model employed for airplane stability and control applications is based upon small flow perturbations from an equilibrium point. The small-disturbance approximation also results in the concept of linear stability derivatives, irrespective of the flow regime in which

Fig. 13.31. Attitude response of a fighter aircraft to a large initial roll-rate disturbance.
the equilibrium point is located. Using the plane of symmetry existing in all atmospheric vehicles, one can separate the rotational motion in the plane of symmetry, called longitudinal dynamics, from that outside the plane referred to as lateral dynamics. A further assumption of de-coupled longitudinal and lateral modes enables a linearized stability analysis commonly applied to airplanes. The attitude control of non-aerodynamic missiles and launch vehicles primarily involves a control torque produced by thrust vectoring, generally leading to a statically unstable configuration. For modern fighter airplanes and missiles, the concentration of mass in the fuselage results in a large difference between the rolling moment of inertia and the pitching (and yawing) inertia. Consequently, the nonlinear, inertial coupling terms in Euler’s equations become significant, thereby causing an interaction between the longitudinal and lateral dynamics for even moderate body rates. Hence, a complete six-degree-of-freedom modeling and simulation become necessary for stability and control analysis of inertia-coupled vehicles.

Exercises

13.1. Calculate the principal inertia tensor, $J_p$, and the principal rotation matrix, $C_p$, for a spacecraft with the following inertia tensor:
Use the result to find the angular velocity in the principal frame if the angular velocity in the current body frame is $\omega = (0.15, -0.25, -0.8)^T \text{ rad/s}$.

13.2. Using the kinematic equations, Eq. (13.40), show that the precession of an axisymmetric spacecraft obeys the relationship $\tan \phi = \frac{\omega_x}{\omega_y}$.

13.3. Write a program to carry out direct numerical simulation of a bang-bang, impulsive attitude maneuver of the spacecraft with the same moments of inertia as in Example 13.3, but with a spin rate of $n = 0.1 \text{ rad/s}$, and realistic thruster torque impulses of magnitude $1000 \text{ N.m}$, each applied for $\Delta t = 0.01 \text{ s}$. What is the time between the impulses and the final deviation of the spin axis?

13.4. Repeat Exercise 13.3 using a reaction wheel spinning about $oy$ instead of the attitude thrusters. The moments of inertia of the wheel in the spacecraft principal axes are $J_{xx} = J_{zz} = 50 \text{ kg.m}^2$ and $J_{yy} = 150 \text{ kg.m}^2$. Consider the reaction wheel to be initially at rest relative to the spacecraft. What is the final spin rate of the wheel at the end of the maneuver?

13.5. It is desired to exactly null the final angular velocity of the spacecraft in Example 13.4 by using thruster torque impulses. Design a maneuver that achieves this using linearized Euler’s equations with small angular velocity approximation. Determine the smallest number and magnitude of the torque impulses if thruster firing is limited to $0.01 \text{ s}$.

13.6. Carry out the attitude simulation of the VSCMG-equipped spacecraft in Example 13.5 using the modified Rodrigues’ parameters (MRP) defined in Chapter 2, and compare the principal rotation angle with that plotted in Fig. 13.16.

13.7. Derive the governing equations of motion for a rigid, asymmetric spacecraft equipped with two reaction wheels, having their spin axes along the major and minor axes of the spacecraft, respectively. Modify the program $spacevscmg.m$ to simulate the response of the spacecraft in Example 13.2 with the two reaction wheels of equal moment of inertia of $10 \text{ kg.m}^2$ about their spin axes. Assume that the wheels are initially at rest relative to the spacecraft, whose initial attitude and angular velocity are specified in Example 13.2. At time $t = 0$, a torque of $10 \text{ N.m}$ begins acting on each wheel about the spin axis, and remains constant for a period of $10 \text{ s}$, after which it instantaneously drops to zero. Neglect friction in the reaction wheel bearings.

13.8. Write a program to simulate the response of an axisymmetric spacecraft platform with an oblate rotor in dual-spin configuration. The spacecraft has
a moment of inertia of 1000 kg\(\cdot\)m\(^2\) about its spin axis, and 2000 kg\(\cdot\)m\(^2\) about a lateral principal axis. The rotor’s moment of inertia about its spin axis is 250 kg\(\cdot\)m\(^2\) and 100 kg\(\cdot\)m\(^2\) about a lateral principal axis. The centers of mass of the platform and the rotor are offset from the center of mass of the dual-spin configuration by 0.5 m and 2.5 m, respectively. Initially, both the platform and the rotor are spinning in the same direction with angular speeds of \(7.27 \times 10^{-5}\) rad/s and \(5\) rad/s, respectively, when a lateral angular velocity disturbance of \(\omega_{xy} = 0.01\) rad/s, is encountered. Neglect the friction in the rotor bearing.

13.9. Estimate the natural frequencies of gravity gradient motion of the \textit{Seasat} spacecraft with the following characteristics:

\[
\begin{align*}
J_{xx} &= J_{yy} = 25, 100 \text{ kg}\cdot\text{m}^2, \\
J_{zz} &= 3000 \text{ kg}\cdot\text{m}^2, \\
n &= 0.00105 \text{ rad/s}.
\end{align*}
\]

Simulate the coupled nonlinear response of the spacecraft to an initial yaw-rate disturbance of \(10^{-5}\) rad.

13.10. Find the phugoid and short-period characteristics of the delta-winged fighter of Example 13.7 if the equilibrium flight path is the straight and level flight of Case (a). How do the settling times compare with those observed in the simulated response of Example 13.7?

13.11. Find the phugoid and short-period characteristics of the jet transport of Example 13.8 if the equilibrium flight path is a quasi-steady climb at standard sea level with \(\delta^e = 45^\circ\), \(v^e = 150\) m/s, and \(\phi^e = 30^\circ\).

13.12. Simulate the longitudinal response of the re-entry vehicle of Example 13.10 to an initial angle of attack disturbance, \(\alpha = 0.002\) rad. What are the phugoid and short-period characteristics of the vehicle?

13.13. Model the combined lateral and longitudinal dynamics of the re-entry vehicle of Example 13.10 using the quaternion instead of Euler angles. Use the model to repeat the simulation of Example 13.10 with the given initial condition, except that the initial pitch angle is \(\Theta^e = -90^\circ\). Will there be any change in the pitch angle during the resulting motion? Why?

13.14. The aerodynamic data for the \textit{F-94} fighter [46] with \(m = 6178.15\) kg, \(J_{xx} = 15,004.5\) kg\(\cdot\)m\(^2\), \(J_{zz} = 455.75\) kg\(\cdot\)m\(^2\), \(J_{zz} = 50,066.26\) kg\(\cdot\)m\(^2\), \(b = 11.37\) m, \(S = 22.22\) m\(^2\), \(A^e = 0\), at Mach number \(M = 0.797\) and standard sea level are the following:

\[
\begin{align*}
C_{n,\alpha} &= 0.1/\text{rad,} \\
C_{n,\alpha} &= -0.134/\text{rad},
\end{align*}
\]
\[ C_{y_b} = -0.546/\text{rad}, \]
\[ C_{y_c} = 0.287/\text{rad}, \]
\[ C_{t_p} = -0.39, \]
\[ C_{t_a} = -0.0654/\text{rad}, \]
\[ C_{t_c} = 0.043, \]
\[ C_L = 0.0605. \]

Simulate the lateral motion of the fighter to an initial sideslip of 0.01 rad from straight and level flight, and identify its lateral modes.

**13.15.** Repeat the simulation of the *Vanguard* missile dynamics of Example 13.11 using the quaternion for attitude representation.